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Correlation functions in a one-dimensional Bose gas

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Abstract. The problem of calculating correlation functions is considered for the one-dimensional Bose gas with a delta-function repulsive interaction between particles. The method of calculation based on the algebraic Bethe ansatz is given.

1. Introduction

The development of the quantum inverse scattering method (QISM) [1] has resulted in a great advance in the investigation of integrable quantum systems in two spacetime dimensions. The important success of the method was the scheme of the algebraic Bethe ansatz [1, 2] which generalises the Bethe method [3] of solving the Heisenberg magnet. It is the thorough analysis of the structure of integrable models implied by the Bethe ansatz that allows the effective approach to calculation of correlation functions. In [4, 5] this approach was applied to the equal-time two-point correlator of currents in the one-dimensional Bose gas which has led to the essential progress in studying this correlator, namely the long-distance asymptotics at non-zero [6] and at zero temperature [7] was obtained.

In this paper the approach of [4, 5] is generalised to include also the two-point correlator of fields as well as any many-point correlator of fields and currents. We hope to use the results obtained here to study the long-distance asymptotics of the correlators in subsequent papers.

In this paper the equal-time correlation functions in the non-relativistic one-dimensional Bose gas with repulsive delta-function interaction between particles is considered. This model was introduced and solved by Lieb and Liniger [8]. It can be considered as a result of second quantisation of the non-linear Schrödinger (NS) equation and is also called the NS model. The corresponding Hamiltonian is

$$H=\int_0^L \mathrm{d}x(\partial_x\psi^+\partial_x\psi+c\psi^+\psi^+\psi\psi-h\psi^+\psi). \tag{1.1}$$

Here $c>0$ is a coupling constant, $h>0$ is a chemical potential, L is a length of a 'box', $\psi(x)$ is a canonical Bose field: $[\psi(x), \psi^+(y)]=\delta(x-y)$. The periodic boundary conditions are supposed to be imposed. The Fock (bare) vacuum $|0\rangle(\psi(x)|0\rangle=0)$ is not the ground state of the Hamiltonian. The ground state $|\Omega\rangle$ is constructed as the Dirac sea over the bare vacuum. The equal-time correlators investigated below are the mean values of the local operator products with respect to the normalised ground-state eigenfunction. The simplest of them are the two-point field correlator $\langle\psi(x)\psi^+(y)\rangle\equiv\langle\Omega|\psi(x)\psi^+(y)|\Omega\rangle$ and the two-point current correlator $\langle j(x)j(y)\rangle\equiv\langle\Omega|j(x)j(y)|\Omega\rangle$ $(j(x)=\psi^+(x)\psi(x))$. The definition of many-point correlators is also quite evident.

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The NS model is the first model to which QISM was applied [1, 9]. It is essential in our approach that one can consider this model as a representative of a class of Bethe ansatz solvable models in QISM possessing the same structure of quantum 'action-angle' variables (i.e. the same quantum R matrix). These models can be parametrised by functional parameters. So the generalised model is introduced where these parameters are considered as arbitrary functions. The dependence of correlators on these can be described explicitly which results in useful representations for the correlation function.

To explain this by the simplest example and to introduce the necessary notations, let us consider the 'one-site' generalised model. The main quantity is the monodromy matrix [1] of the model. It is the 2×2 matrix $T(\lambda)$ depending on the complex spectral parameter λ

$$T(\lambda)=\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda)\end{pmatrix}. \tag{1.2}$$

Matrix elements A, B, C, D are operators acting in 'quantum' space $\mathscr{H}$. Their commutation relations are given by the 4×4 matrix $R(\lambda, \mu)$ with c-number matrix elements

$$R(\lambda,\mu)T(\lambda)\otimes T(\mu)=T(\mu)\otimes T(\lambda)R(\lambda,\mu) \tag{1.3}$$

$$R(\lambda,\mu)=\begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda)\end{pmatrix} \tag{1.4}$$

where

$$f(\lambda,\mu)=(\lambda-\mu+\mathrm{i}c)/(\lambda-\mu) \qquad g(\lambda,\mu)=\mathrm{i}c/(\lambda-\mu). \tag{1.5}$$

We suppose also that the bare vacuum $|0\rangle$ exists. This is the vector in $\mathscr{H}$ with the following properties:

$$\begin{aligned} &A(\lambda)|0\rangle=a(\lambda)|0\rangle \qquad D(\lambda)|0\rangle=d(\lambda)|0\rangle \\ &C(\lambda)|0\rangle=0 \qquad B(\lambda)|0\rangle\neq 0. \end{aligned} \tag{1.6}$$

The dual vacuum $\langle 0|$ is defined by the requirements that

$$\begin{aligned} &\langle 0|A(\lambda)=a(\lambda)\langle 0| \qquad \langle 0|D(\lambda)=d(\lambda)\langle 0| \\ &\langle 0|B(\lambda)=0 \qquad \langle 0|0\rangle=1. \end{aligned} \tag{1.7}$$

Monodromy matrices with properties (1.2)-(1.6) do exist for arbitrary c-number functions $a(\lambda)$ and $d(\lambda)$ [10] which can thus be considered as free functional parameters. Essentially different monodromy matrices are parametrised by different functions $r(\lambda)$

$$r(\lambda)=a(\lambda)/d(\lambda). \tag{1.8}$$

Consider now vectors $\Pi_{j=1}^{N}\mathbb{B}(\lambda_j)|0\rangle$ and covectors $\langle 0|\Pi_{j=1}^{N}\mathbb{C}(\lambda_j)$ where the following normalisation of 'creation' and 'annihilation' operators is used:

$$\mathbb{B}(\lambda)\equiv B(\lambda)/d(\lambda) \qquad \mathbb{C}(\lambda)\equiv C(\lambda)/d(\lambda). \tag{1.9}$$

Their scalar products can be calculated by means of the commutation relations (1.3) and properties (1.5) and (1.6). The following representation is obtained in [11] for scalar products (λ_l^C, λ_k^B below are arbitrary different complex numbers):

$$\left\langle 0 \left| \prod_{l=1}^{M} \mathbb{C}(\lambda_l^C) \prod_{k=1}^{N} \mathbb{B}(\lambda_k^B) \right| 0 \right\rangle = \delta_{MN} \sum_{\text{part}} \prod_{\lambda^{AB}} r(\lambda^{AB})$$
$$\times \prod_{\lambda^{AC}} r(\lambda^{AC}) \prod_{\lambda^{AC}} \prod_{\lambda^{DC}} f(\lambda^{AC}, \lambda^{DC}) \prod_{\lambda^{DB}} \prod_{\lambda^{AB}} f(\lambda^{AB}, \lambda^{DB})$$
$$\times Z_n(\{\lambda^{DB}\}_n, \{\lambda^{AC}\}_n) Z_{N-n}(\{\lambda^{DC}\}_{N-n}, \{\lambda^{AB}\}_{N-n}). \tag{1.10}$$

The sum here is taken over all the partitions of the initial set

$$\{\lambda_l^C;\, l=1,\ldots,N\}_N \cup \{\lambda_k^B;\, k=1,\ldots,N\}_N = \{\lambda^A\}_N \cup \{\lambda^D\}_N$$

into four subsets:

$$\{\lambda^{AC}\}_n = \{\lambda^A\}_N \cap \{\lambda^C\}_N \qquad \{\lambda^{AB}\}_{N-n} = \{\lambda^A\}_N \cap \{\lambda^B\}_N$$
$$\{\lambda^{DB}\}_n = \{\lambda^D\}_N \cap \{\lambda^B\}_N \qquad \{\lambda^{DC}\}_{N-n} = \{\lambda^D\}_N \cap \{\lambda^C\}_N.$$

Subscript m in $\{\lambda\}_m$ denotes the number of elements in the set. The dependence on arbitrary function $r(\lambda)$ is written explicitly in (1.10). Functions $Z_n(\{\lambda\}_n, \{\mu\}_n)$ do not depend on $r(\lambda)$. Their definition, as well as recursion formulae permitting the calculation of any of them, are given in appendix 1.

The transfer matrix $\tau(\lambda) \equiv A(\lambda) + D(\lambda)$ generates the Hamiltonians of integrable systems. Operators $\tau(\lambda)$ at different λ commute: $[\tau(\lambda), \tau(\mu)] = 0$; so eigenvectors of $\tau(\lambda)$ do not depend on λ. Bethe's eigenvectors $|\lambda_1, \ldots, \lambda_N\rangle$ and corresponding covectors $\langle\lambda_1, \ldots, \lambda_N|$ are of the form

$$|\lambda_1, \ldots, \lambda_N\rangle = \prod_{j=1}^{N} \mathbb{B}(\lambda_j)|0\rangle \qquad \langle\lambda_1, \ldots, \lambda_N| = \langle 0| \prod_{j=1}^{N} \mathbb{C}(\lambda_j) \tag{1.11}$$

where spectral parameters λ_j are different and satisfy the system of generalised Bethe equations

$$r_j \equiv r(\lambda_j) = \prod_{k \neq j} (f(\lambda_k, \lambda_j)/f(\lambda_j, \lambda_k)). \tag{1.12}$$

The corresponding eigenvalue of $\tau(\lambda)$ is

$$\tau(\lambda)|\lambda_1, \ldots, \lambda_N\rangle = t_N(\lambda; \lambda_1, \ldots, \lambda_N)|\lambda_1, \ldots, \lambda_N\rangle$$
$$\langle\lambda_1, \ldots, \lambda_N|\tau(\lambda) = t_N(\lambda; \lambda_1, \ldots, \lambda_N)\langle\lambda_1, \ldots, \lambda_N| \tag{1.13}$$
$$t_N = a(\lambda) \prod_{j=1}^{N} f(\lambda, \lambda_j) + d(\lambda) \prod_{j=1}^{N} f(\lambda_j, \lambda).$$

The 'norm' of Bethe's eigenvectors can be obtained from (1.10) by taking the limit $\lambda_j^B \to \lambda_j^C \to \lambda_j$ $(j = 1, \ldots, N)$ and imposing the system (1.12) on λ_j. The result is [11]

$$\langle\lambda_1, \ldots, \lambda_N|\lambda_1, \ldots, \lambda_N\rangle = c^N \left(\prod_{\substack{j,k=1 \\ j \neq k}}^{N} f(\lambda_j, \lambda_k) \right) \det_N(\varphi'). \tag{1.14}$$

The $N \times N$ matrix φ' here is defined as (compare with (1.12))

$$(\varphi')_{jk} = \partial\varphi_j/\partial\lambda_k$$
$$\varphi_j \equiv \varphi(\lambda_j) = \mathrm{i} \log r_j + \mathrm{i} \sum_{k \neq j}^{N} \log(f(\lambda_j, \lambda_k)/f(\lambda_k, \lambda_j)). \tag{1.15}$$

We now discuss the relation of the NS model (1.1) to the generalised model considered above. It is convenient to use the regularised version of the model, namely the lattice NS model introduced in [12]. The monodromy matrix of this model is the product of local L operators located at the sites of the one-dimensional space lattice

$$T(\lambda)=L_M(\lambda)L_{M-1}(\lambda)\dots L_2(\lambda)L_1(\lambda) \tag{1.16}$$

where

$$L_n(\lambda)=\begin{pmatrix} 1-\frac{1}{2}\mathrm{i}\lambda\Delta+\frac{1}{2}c\Delta\psi_n^+\psi_n & -\mathrm{i}(c\Delta)^{1/2}\psi_n^+(1+\frac{1}{4}c\Delta\psi_n^+\psi_n)^{1/2} \\ \mathrm{i}(c\Delta)^{1/2}(1+\frac{1}{4}c\Delta\psi_n^+\psi_n)^{1/2}\psi_n & 1+\frac{1}{2}\mathrm{i}\lambda\Delta+\frac{1}{2}c\Delta\psi_n^+\psi_n \end{pmatrix}. \tag{1.17}$$

Here ψ_n are canonical Bose variables

$$[\psi_n,\psi_m^+]=\delta_{mn} \qquad [\psi_n,\psi_m]=[\psi_n^+,\psi_m^+]=0. \tag{1.18}$$

The commutation relations between the matrix elements of the L operator are of the same form as those of the monodromy matrix (2.2). The quantum space $\mathscr{H}$ is a tensor product of the local Fock spaces $\mathfrak{H}_n$ over all the sites

$$\mathscr{H}=\bigotimes_{n=1}^{M}\mathfrak{H}_n. \tag{1.19}$$

The Fock vacuum $|0\rangle$ is the tensor product of the local Fock vacua ω_n: $|0\rangle=\otimes_1^M\omega_n$; $\psi_n\omega_n=0$. Functions $a(\lambda)$, $d(\lambda)$ and $r(\lambda)$ are

$$a(\lambda)=(1-\tfrac{1}{2}\mathrm{i}\lambda\Delta)^M \qquad d(\lambda)=(1+\tfrac{1}{2}\mathrm{i}\lambda\Delta)^M \qquad r(\lambda)=a(\lambda)/d(\lambda).$$

The monodromy matrix of the NS model (1.1) is obtained from $T(\lambda)$ (2.5) by taking the limit $\Delta\to 0$, $M\to\infty$, $M\Delta=L$; one has then the correspondence $\psi_n\simeq(\int_{X_n}^{X_{n+1}}\psi(x)\,\mathrm{d}x)\Delta^{-1/2}$; $x_n=n\Delta$. The function $r(\lambda)$ in this limit is

$$r_{\mathrm{NS}}(\lambda)=\exp(-\mathrm{i}L\lambda). \tag{1.20}$$

The Hamiltonian (1.1) can be expressed in terms of the transfer matrix [1]. Thus eigenfunctions of the Hamiltonian are given by (1.11) and (1.12).

The generalised model considered above was called the one-site model because the structure of the monodromy matrix $T(\lambda)$ is not specified in (1.2)-(1.15). Nevertheless the existence of an arbitrary functional parameter had resulted in calculating the 'norm' of the Bethe wavefunction (1.14) and (1.15). To calculate correlation functions, however, the generalised models with more detailed structure of monodromy matrices have to be considered.

2. Two-site generalised model

We begin with the construction of the two-site model permitting the calculation of the field correlator in the NS model (1.1). Consider first the monodromy matrix (1.2) with the following structure (for L_n see (1.17)):

$$T(\lambda)=T_2(\lambda)L_n(\lambda)T_1(\lambda)L_1(\lambda). \tag{2.1}$$

The quantum space where the matrix elements of $T(\lambda)$ act is $\mathscr{H}=\mathscr{H}_2\otimes\mathfrak{H}_n\otimes\mathscr{H}_1\otimes\mathfrak{H}_1$. Each of the four factors at the right-hand side of (2.1) acts non-trivially only in its own quantum space and acts as the unit operator in other spaces. Commutation relations of matrix elements of $T_i(\lambda)$ ($i=1,2$) (these matrix elements are denoted as

A_i, B_i, C_i, D_i; cf (1.2)) are given by the same formula (1.3). It is also supposed that the monodromy matrices $T_i(\lambda)$ $(i=1,2)$ possess the bare vacua $|0\rangle_i$

$$A_i(\lambda)|0\rangle_i = a_i(\lambda)|0\rangle_i \qquad D_i(\lambda)|0\rangle_i = d_i(\lambda)|0\rangle_i$$
$$C_i(\lambda)|0\rangle_i = 0 \qquad B_i(\lambda)|0\rangle_i \neq 0. \tag{2.2}$$

Hence the state vector $|0\rangle$

$$|0\rangle = |0\rangle_2 \otimes \omega_n \otimes |0\rangle_1 \otimes \omega_1 \tag{2.3}$$

is the bare vacuum for $T(\lambda)$ ((1.6) and (1.7)) with functions $a(\lambda)$ and $d(\lambda)$ given by

$$a(\lambda) = a_1(\lambda)a_2(\lambda)(1-\tfrac{1}{2}\mathrm{i}\lambda\Delta)^2 \qquad d(\lambda) = d_1(\lambda)d_2(\lambda)(1+\tfrac{1}{2}\mathrm{i}\lambda\Delta)^2. \tag{2.4}$$

All the relations (1.2)-(1.15) are valid also in the two-site model. It is also quite obvious that the monodromy matrix (1.16) is a particular case of the generalised model corresponding to

$$a_1(\lambda) = (1-\tfrac{1}{2}\mathrm{i}\lambda\Delta)^{M-n-1} \qquad a_2(\lambda) = (1-\tfrac{1}{2}\mathrm{i}\lambda\Delta)^{n-1} \qquad d_i(\lambda) = a_i^*(\lambda^*).$$

It is, however, of importance that functions $a_i(\lambda)$, $d_i(\lambda)$ $(i=1,2)$ in the generalised model can be considered as arbitrary functional parameters. This appears to be essential in the investigation of correlation functions.

Further we are interested in correlation functions in the generalised model at the continuous limit where $\Delta \to 0$, $n\Delta \to x$, $M\Delta \to L$ (matrices $T_i(\lambda)$ $(i=1,2)$ remain arbitrary monodromy matrices with the properties described above). At this limit $L_n(\lambda) = 1-\mathrm{i}\mathscr{L}(x)\Delta$, where $\mathscr{L}(x)$ is the continuous L operator [1] and one has for the monodromy matrix (2.1)

$$T(\lambda) = T_2(\lambda)T_1(\lambda). \tag{2.5}$$

Function $r(\lambda)$ in (1.8) is now

$$r(\lambda) = l(\lambda)m(\lambda) \tag{2.6}$$

where arbitrary functions $l(\lambda)$ and $m(\lambda)$ are

$$l(\lambda) = a_1(\lambda)/d_1(\lambda) \qquad m(\lambda) = a_2(\lambda)/d_2(\lambda). \tag{2.7}$$

The commutators of lattice fields ψ_1, ψ_1^+, ψ_n, ψ_n^+ with the matrix elements of $T(\lambda)$ do not, however, vanish at the continuous limit. Using (1.3), (1.17), (1.18) and (2.1) one obtains at $\Delta \to 0$ for fields $\psi(x)$, $\psi^+(x)$ entering (1.1)

$$[\psi(0), T(\lambda)] \doteq -\mathrm{i}c^{1/2}\begin{pmatrix} 0 & A(\lambda) \\ 0 & C(\lambda) \end{pmatrix} \tag{2.8}$$

$$[T(\lambda), \psi^+(0)] = -\mathrm{i}c^{1/2}\begin{pmatrix} B(\lambda) & 0 \\ D(\lambda) & 0 \end{pmatrix} \tag{2.9}$$

$$[\psi(x), T(\lambda)] = -\mathrm{i}c^{1/2}\begin{pmatrix} A_2(\lambda)C_1(\lambda) & A_2(\lambda)D_1(\lambda) \\ C_2(\lambda)C_1(\lambda) & C_2(\lambda)D_1(\lambda) \end{pmatrix} \tag{2.10}$$

$$[T(\lambda), \psi^+(x)] = -\mathrm{i}c^{1/2}\begin{pmatrix} B_2(\lambda)A_1(\lambda) & B_2(\lambda)B_1(\lambda) \\ D_2(\lambda)A_1(\lambda) & D_2(\lambda)B_1(\lambda) \end{pmatrix}. \tag{2.11}$$

Using commutation relations (1.3) one can express [4] vectors $\Pi_j \mathbb{B}(\lambda_j)|0\rangle$ and $\langle 0|\Pi_j \mathbb{C}(\lambda_j)$ in terms of vectors $\Pi_j \mathbb{B}_i(\lambda_j)|0\rangle$ and $\langle 0|\Pi_j \mathbb{C}_i(\lambda_j)$ $(i=1,2)$ (operators $\mathbb{B}_i$, $\mathbb{C}_i$ are defined similar to (1.9) as $B_i(\lambda)=B_i(\lambda)/d_i(\lambda)$ and $\mathbb{C}_i(\lambda)=C_i(\lambda)/d_i(\lambda)$)

$$\prod_{j=1}^{N} \mathbb{B}(\lambda_j)|0\rangle = \sum_{\substack{\{\lambda\}=\{\lambda^{(1)}\}\cup\{\lambda^{(2)}\}\\ \{\lambda^{(1)}\}\cap\{\lambda^{(2)}\}=\varnothing}} \prod_{\lambda\in\{\lambda^{(1)}\}} m(\lambda)\mathbb{B}_1(\lambda) \prod_{\mu\in\{\lambda^{(2)}\}} f(\lambda,\mu)\mathbb{B}_2(\mu)|0\rangle \tag{2.12}$$

$$\langle 0|\prod_{j=1}^{N} \mathbb{C}(\lambda_j) = \sum_{\substack{\{\lambda\}=\{\lambda^{(1)}\}\cup\{\lambda^{(2)}\}\\ \{\lambda^{(1)}\}\cap\{\lambda^{(2)}\}=\varnothing}} \prod_{\lambda\in\{\lambda^{(2)}\}} l(\lambda)\langle 0|\mathbb{C}_2(\lambda) \prod_{\mu\in\{\lambda^{(1)}\}} f(\lambda,\mu)\mathbb{C}_1(\mu). \tag{2.13}$$

Equations (2.8)-(2.13) result in the following formulae which appear to be very useful:

$$\psi(0)\prod_{j=1}^{N} \mathbb{B}(\lambda_j)|0\rangle = -\mathrm{i}c^{1/2}\sum_{j=1}^{N} r(\lambda_j)\prod_{k\neq j} f(\lambda_j,\lambda_k)\prod_{k\neq j}\mathbb{B}(\lambda_k)|0\rangle \tag{2.14}$$

$$\begin{aligned}\psi(x)\prod_{j=1}^{N} \mathbb{B}(\lambda_j)|0\rangle = -\mathrm{i}c^{1/2} &\sum_{\{\lambda\}=\{\lambda^{(1)}\}\cup\{\lambda^{(2)}\}\cup\{\lambda^{(0)}\}} l(\lambda^{(0)})\\ &\times\prod_{\lambda^{(1)}} m(\lambda^{(1)})\prod_{\lambda^{(2)}} f(\lambda^{(1)},\lambda^{(2)})f(\lambda^{(0)},\lambda^{(2)})\\ &\times\prod_{\lambda^{(1)}}\mathbb{B}_1(\lambda^{(1)})\prod_{\lambda^{(2)}}\mathbb{B}_2(\lambda^{(2)})|0\rangle\end{aligned} \tag{2.15}$$

$$\langle 0|\prod_{j=1}^{N}\mathbb{C}(\lambda_j)\psi^+(0) = \mathrm{i}c^{1/2}\sum_{j=1}^{N}\prod_{k\neq j} f(\lambda_k,\lambda_j)\langle 0|\prod_{k\neq j}\mathbb{C}(\lambda_k) \tag{2.16}$$

$$\begin{aligned}\langle 0|\prod_{j=1}^{N}\mathbb{C}(\lambda_j)\psi^+(x) = \mathrm{i}c^{1/2} &\sum_{\{\lambda\}=\{\lambda^{(1)}\}\cup\{\lambda^{(2)}\}\cup\{\lambda^{(0)}\}} l(\lambda^{(0)})\prod_{\lambda^{(1)}} f(\lambda^{(0)},\lambda^{(1)})\\ &\times\prod_{\lambda^{(2)}} l(\lambda^{(2)})f(\lambda^{(2)},\lambda^{(0)})f(\lambda^{(2)},\lambda^{(1)})\langle 0|\prod_{\lambda^{(1)}}\mathbb{C}_1(\lambda^{(1)})\prod_{\lambda^{(2)}}\mathbb{C}_2(\lambda^{(2)}).\end{aligned} \tag{2.17}$$

Formulae (2.14)-(2.17) can be considered as the definition of field operators in the two-site generalised model. If one requires that conditions $d_i^*(\lambda^*)=a_i(\lambda)$ and $C_i^+(\lambda)=-B_i(\lambda)$ are satisfied (as in the NS model) then relations (2.16) and (2.17) can be obtained by conjugation of (2.14) and (2.15).

3. Field form factor

The field form factor is a matrix element $\mathcal{F}_N$ of the field operator $\psi(0)$ between Bethe eigenvectors (1.11) of the transfer matrix

$$\mathcal{F}_N(\{\lambda^C\}_{N-1},\{\lambda^B\}_N) = \langle\lambda_1^C\ldots\lambda_{N-1}^C|\psi(0)|\lambda_1^B\ldots\lambda_N^B\rangle. \tag{3.1}$$

It should be emphasised that spectral parameters entering the sets $\{\lambda^C\}_{N-1}$ and $\{\lambda^B\}_N$ satisfy the corresponding system of Bethe equations (1.12). Due to the translational invariance the matrix element of the operator $\psi(x)$ is

$$\langle\{\lambda^C\}_{N-1}|\psi(x)|\{\lambda^B\}_N\rangle = \prod_{j=1}^{N-1} l(\lambda_j^C)\prod_{k=1}^{N} l^{-1}(\lambda_k^B)\langle\{\lambda^C\}_{N-1}|\psi(0)|\{\lambda^B\}_N\rangle. \tag{3.2}$$

To obtain an explicit representation for the form factor (3.1), one makes use of (2.14), (2.13), (1.10) and (1.12). The result is

$$\begin{aligned}\mathscr{F}_N(\{\lambda^C\}_{N-1},\{\lambda^B\}_N) = -\mathrm{i}c^{1/2} \sum_{\substack{\{\lambda^C\}_{N-1}\cup\{\lambda^B\}_N \\ =\{\lambda^A\}_{N-1}\cup\{\lambda^D\}_{N-1}\cup\{\lambda^0\}_1}} \\ \times \prod_{\lambda^{AB}} f(\lambda^0,\lambda^{AB}) \prod_{\lambda^{DB}} f(\lambda^{DB},\lambda^0) \prod_{\lambda^{DC}} \prod_{\lambda^{AC}} f(\lambda^{DC},\lambda^{AC}) \\ \times \prod_{\lambda^{DB}} \prod_{\lambda^{AB}} f(\lambda^{DB},\lambda^{AB}) Z_n(\{\lambda^{DB}\}_n,\{\lambda^{AC}\}_n) \\ \times Z_{N-n-1}(\{\lambda^{DC}\}_{N-n-1},\{\lambda^{AB}\}_{N-n-1}).\end{aligned} \tag{3.3}$$

The dependence of the form factor on parameters of the generalised model (i.e. on functions $r(\lambda)$, $l(\lambda)$, $m(\lambda)$ defined by (2.6) and (2.7)) is only due to the fact that spectral parameters λ_j^C, λ_k^B in (3.3) are solutions of the Bethe systems (1.12) for $\{\lambda^C\}_{N-1}$ and $\{\lambda^B\}_N$. So one can make the analytical continuation and consider $\mathscr{F}_N$ as a function of $(2N-1)$ independent complex variables λ_j^C, λ_k^B, this function being defined only by the R matrix (1.4) and (1.5). It can be easily proved that this function is symmetric in $\lambda_j^C \in \{\lambda^C\}_{N-1}$ and $\lambda_k^B \in \{\lambda^B\}_N$.

To calculate functions $\mathscr{F}_N$ the following recursive procedure can also be used. Equation (3.3) and the recursion formula for coefficients Z given in appendix 1 show that the principal singularity of $\mathscr{F}_N$ at $\lambda_j^C \to \lambda_k^B$ is the first-order pole, the residue at this pole being proportional to $\mathscr{F}_{N-1}$. If $\lambda_1^B \to \lambda_1^C$ one has (the general case is obvious due to the symmetry mentioned above)

$$\begin{aligned}\mathscr{F}_N(\{\lambda^C\}_{N-1},\{\lambda^B\}_N)|_{\lambda_1^C\to\lambda_1^B} \to \frac{\mathrm{i}c}{(\lambda_1^B-\lambda_1^C)}\left(\prod_{j=2}^{N-1} f(\lambda_1^C,\lambda_j^C)\prod_{j=2}^{N} f(\lambda_j^B,\lambda_1^B)\right. \\ \left. -\prod_{j=2}^{N-1} f(\lambda_j^C,\lambda_1^C)\prod_{j=2}^{N} f(\lambda_1^B,\lambda_j^B)\right)\mathscr{F}_{N-1}(\{\lambda^C\}_{N-2},\{\lambda^B\}_{N-1})+\ldots.\end{aligned} \tag{3.4}$$

Here terms regular at $\lambda_1^B \to \lambda_1^C$ are not written down. Extracting all the singularities of this kind, one can represent $\mathscr{F}_N$ as follows:

$$\mathscr{F}_N(\{\lambda^C\}_{N-1},\{\lambda^B\}_N) = \mathscr{P}_N(\{\lambda^C\}_{N-1},\{\lambda^B\}_N)\left(\prod_{k=1}^{N-1}\prod_{j=1}^{N}(\lambda_k^B-\lambda_j^C)\right)^{-1}. \tag{3.5}$$

It is shown in appendix 2 that (i) functions $\mathscr{P}_N$ are polynomials in λ and (ii) relations (3.4) permit the calculation of these polynomials recursively in N beginning with $N=1$. The first three polynomials are

$$\mathscr{P}_1 = -\mathrm{i}c^{1/2} \tag{3.6}$$

$$\mathscr{P}_2 = -2\mathrm{i}c^{1/2}(\mathrm{i}c)^2 \tag{3.7}$$

$$\begin{aligned}\mathscr{P}_3 = -4\mathrm{i}c^{1/2}\{(\mathrm{i}c)^6+(\mathrm{i}c)^4[(\lambda_1^B-\lambda_1^C)(\lambda_2^B-\lambda_2^C) \\ +(\lambda_1^B-\lambda_2^C)(\lambda_3^B-\lambda_1^C)+(\lambda_2^B-\lambda_1^C)(\lambda_3^B-\lambda_2^C)]\}.\end{aligned} \tag{3.8}$$

The asymptotical expansion of $\mathscr{P}_N$ at the strong coupling limit $(c\to\infty)$ is

$$\begin{aligned}\mathscr{P}_N = 2^{N-1}(-\mathrm{i}c^{1/2})(\mathrm{i}c)^{N(N-1)}+2^{N-1}(-\mathrm{i}c^{1/2})(\mathrm{i}c)^{N(N-1)-2} \\ \times[2N-2]^{-1}\sum_{j\neq k,p\neq q}(\lambda_j^B-\lambda_p^C)(\lambda_k^B-\lambda_q^C)+\ldots.\end{aligned} \tag{3.9}$$

Formulae (3.6)-(3.9) can also be obtained directly from the representation (3.3). So we have demonstrated that the two-site generalised model permits investigation of the field form factor (3.1) (or (3.2)).

4. Multi-site model

To investigate form factors of several fields one has to introduce the multi-site generalised model. The number of factors in the decomposition of the monodromy matrix similar to (2.5) will now depend on the number of fields entering the form factor.

The following form factors will be studied:

$$\mathcal{F}_{N_C,N_B}(\{\lambda^C\}_{N_C},\{\lambda^B\}_{N_B}) = \langle\{\lambda^C\}|\varphi_{a_0}(x_0)\mu_{\theta_0}(x_0,x_1)\dots\varphi_{a_A}(x_A)\mu_{\theta_A}(x_A,x_{A+1})|\{\lambda^B\}\rangle \tag{4.1}$$

where $a_i=-1,0,1$; $0\leq x_0<x_1<\dots<x_{A+1}\leq L$ and the spectral parameters $\lambda_j^C\in\{\lambda^C\}_{N_C}$ and $\lambda_k^B\in\{\lambda^B\}_{N_B}$ satisfy the corresponding Bethe systems (1.12). We use the notations

$$\varphi_1(x)=\psi^+(x)\qquad \varphi_{-1}(x)=\psi(x)\qquad \varphi_0(x)=1 \tag{4.2}$$

$$\mu_{\theta_\alpha}(x_\alpha,x_{\alpha+1})=\exp(\theta_\alpha q_{\alpha,\alpha+1}) \tag{4.3}$$

where operators $q_{\alpha,\alpha+1}$ are operators of the number of particles corresponding to the intervals $[x_\alpha,x_{\alpha+1})$ in the NS model

$$q_{\alpha,\alpha+1}=\lim_{\varepsilon\to 0}\int_{x_\alpha}^{x_{\alpha+1}-\varepsilon}\psi^+(x)\psi(x)\,dx.$$

The regularisation in the commutation relations of operators ψ, ψ^+ and μ_θ is chosen in such a way that

$$\begin{aligned}\mu_\theta(x,y)\psi(y)&=\psi(y)\mu_\theta(x,y)\\ \psi(x)\mu_\theta(x,y)&=\mu_\theta(x,y)\psi(x)\exp\theta.\end{aligned} \tag{4.4}$$

Due to the translation invariance one can put $x_0=0$. To study the form factor (4.1) one introduces the $(A+1)$-site generalised model with the following structure of the monodromy matrix (1.2):

$$T(\lambda)=T_{A,A+1}(\lambda)T_{A-1,A}(\lambda)\dots T_{0,1}(\lambda). \tag{4.5}$$

This decomposition of the monodromy matrix corresponds to the decomposition of the interval $[0,L)$ in the NS model (1.1) into intervals $[x_\alpha,x_{\alpha+1})$ for $\alpha=0,\dots,A$ and $x_{A+1}\equiv L$.

Matrix elements of matrices $T_{\alpha,\alpha+1}(\lambda)$ are denoted as $A_{\alpha,\alpha+1}(\lambda)$, $B_{\alpha,\alpha+1}(\lambda)$, $C_{\alpha,\alpha+1}(\lambda)$ and $D_{\alpha,\alpha+1}(\lambda)$. Matrix $T_{\alpha,\alpha+1}$ possesses the 'vacuum' $|0\rangle_{\alpha,\alpha+1}$

$$C_{\alpha,\alpha+1}(\lambda)|0\rangle_{\alpha,\alpha+1}=0 \tag{4.6}$$

$$A_{\alpha,\alpha+1}(\lambda)|0\rangle_{\alpha,\alpha+1}=a_{\alpha,\alpha+1}(\lambda)|0\rangle_{\alpha,\alpha+1}\qquad D_{\alpha,\alpha+1}(\lambda)|0\rangle_{\alpha,\alpha+1}=d_{\alpha,\alpha+1}(\lambda)|0\rangle_{\alpha,\alpha+1}. \tag{4.7}$$

The state $|0\rangle=\bigotimes_0^A|0\rangle_{\alpha,\alpha+1}$ is the vacuum for $T(\lambda)$ (1.6). Arbitrary functions $a_{\alpha,\alpha+1}(\lambda)$ and $d_{\alpha,\alpha+1}(\lambda)$ will be called the parameters of the multi-site model. The formulae similar to (2.12) and (2.13) also exist in the multi-site model

$$\prod_{j=1}^{N_B}\mathbb{B}(\lambda_j^B)|0\rangle=\sum_{\{\lambda^B\}=\bigcup_{\alpha=0}^A\{\lambda_\alpha^B\}}\prod_{\alpha<\beta}^{A}\prod_{\lambda_\alpha^B}l_{\beta,\beta+1}(\lambda_\alpha^B)\prod_{\lambda_\beta^B}f(\lambda_\alpha^B,\lambda_\beta^B)\prod_{\alpha=0}^{A}\prod_{\lambda_\alpha^B}\mathbb{B}_{\alpha,\alpha+1}(\lambda_\alpha^B)|0\rangle \tag{4.8}$$

$$\begin{aligned}\langle 0|\prod_{j=1}^{N_c}\mathbb{C}(\lambda_j^c)=&\sum_{\{\lambda^C\}=\bigcup_{\alpha=0}^A\{\lambda_\alpha^C\}}\prod_{\alpha<\beta}^{A}\prod_{\lambda_\beta^C}l_{\alpha,\alpha+1}(\lambda_\beta^C)\\ &\times\prod_{\lambda_\alpha^C}f(\lambda_\beta^C,\lambda_\alpha^C)\langle 0|\prod_{\alpha=0}^{A}\prod_{\lambda_\alpha^C}\mathbb{C}_{\alpha,\alpha+1}(\lambda_\alpha^C).\end{aligned} \tag{4.9}$$

Operators $\mathbb{B}_{\alpha,\alpha+1}(\lambda)=B_{\alpha,\alpha+1}(\lambda)/d_{\alpha,\alpha+1}(\lambda)$ and $\mathbb{C}_{\alpha,\alpha+1}(\lambda)=C_{\alpha,\alpha+1}(\lambda)/d_{\alpha,\alpha+1}(\lambda)$ are defined similar to (1.9).

Turn now to the form factor (4.1). It can be readily defined in terms of the generalised model described above. The definition of fields $\psi(x_\alpha)$, $\psi^+(x_\alpha)$ ($\alpha=0,\ldots,A$) in the generalised model is quite obvious from § 2, where local fields were introduced in the two-site model. Operators $q_{\alpha,\alpha+1}$ are defined similar to the operator Q_1 of [4]

$$q_{\alpha,\alpha+1}\prod_{j=1}^{N}\mathbb{B}_{\alpha,\alpha+1}(\lambda_j)|0\rangle=N\prod_{j=1}^{N}\mathbb{B}_{\alpha,\alpha+1}(\lambda_j)|0\rangle$$

$$\langle 0|\prod_{j=1}^{N}\mathbb{C}_{\alpha,\alpha+1}(\lambda_j)q_{\alpha,\alpha+1}=N\langle 0|\prod_{j=1}^{N}\mathbb{C}_{\alpha,\alpha+1}(\lambda_j).$$

Commutation relations (4.4) are also valid in the generalised model.

Putting (4.8) into (4.1) one obtains the following representation for $\mathscr{F}$:

$$\mathscr{F}=\sum_{\text{part}}\prod_{\alpha<\beta}\prod_{\lambda_\alpha^B}l_{\beta,\beta+1}(\lambda_\alpha^B)\prod_{\lambda_\beta^B}f(\lambda_\alpha^B,\lambda_\beta^B)\prod_{\alpha<\beta}\prod_{\lambda_\beta^C}l_{\alpha,\alpha+1}(\lambda_\beta^C)\prod_{\lambda_\alpha^C}f(\lambda_\beta^C,\lambda_\alpha^C)$$
$$\times\prod_{\alpha=0}^{A}\left\langle 0\left|\prod_{\lambda_\alpha^C}\mathbb{C}_{\alpha,\alpha+1}(\lambda_\alpha^C)\varphi_{a_\alpha}(x_\alpha)\mu_{\theta_\alpha}(x_\alpha,x_{\alpha+1})\prod_{\lambda_\alpha^B}\mathbb{B}_{\alpha,\alpha+1}(\lambda_\alpha^B)\right|0\right\rangle. \qquad (4.10)$$

The sum here is taken over all the partitions of the set $\{\lambda^B\}_N=\{\lambda_j^B;j=1,\ldots,N\}$ into $(A+1)$ subsets $\{\lambda_\alpha^B\}$ and over all the partitions of the set $\{\lambda^C\}_N$ into $(A+1)$ subsets $\{\lambda_\alpha^C\}$:

$$\{\lambda^B\}=\bigcup_{\alpha=0}^{A}\{\lambda_\alpha^B\}\qquad\{\lambda^C\}=\bigcup_{\alpha=0}^{A}\{\lambda_\alpha^C\}.$$

Matrix elements entering (4.10) can be reduced to scalar products (1.10) and single-field form factors

$$\left\langle 0\left|\prod_{\lambda_\alpha^C}\mathbb{C}_{\alpha,\alpha+1}(\lambda_\alpha^C)\varphi_{a_\alpha}(x_\alpha)\mu_{\theta_\alpha}(x_\alpha,x_{\alpha+1})\prod_{\lambda_\alpha^B}\mathbb{B}_{\alpha,\alpha+1}(\lambda_\alpha^B)\right|0\right\rangle$$
$$=\exp\{\theta_\alpha N^B_{\alpha,\alpha+1}\}\left\langle 0\left|\prod_{\lambda_\alpha^C}\mathbb{C}_{\alpha,\alpha+1}(\lambda_\alpha^C)\varphi_{a_\alpha}(x_\alpha)\prod_{\lambda_\alpha^B}\mathbb{B}_{\alpha,\alpha+1}(\lambda_\alpha^B)\right|0\right\rangle. \qquad (4.11)$$

Further considerations can be approached in two different ways. The first way is to use the explicit representations for scalar products (1.10) and for the form factors of fields $\psi(x)$, $\psi^+(x)$, (3.3) and (3.2), which give the representation of the form factor (4.1) in terms of the parameters of the generalised model as well as of functions $Z(\{\lambda\},\{\mu\})$ described in appendix 1. The second way is to determine the dependence of $\mathscr{F}$ (4.1) on the parameters of the generalised model by studying singularities of $\mathscr{F}$ (this method was used in [4] to calculate the form factor of currents). Both methods give, of course, the same representation for the form factor (4.1)

$$\mathscr{F}_{N_CN_B}(\{\lambda^C\},\{\lambda^B\})=\sum_{\text{part}}\prod_{\alpha=1}^{A}\prod_{\lambda_\alpha^C}l_\alpha(\lambda_\alpha^C)\prod_{\lambda_\alpha^B}l_\alpha^{-1}(\lambda_\alpha^B)$$
$$\times\prod_{\alpha>\beta}\prod_{\lambda_\alpha^C}\prod_{\lambda_\beta^C}f(\lambda_\alpha^C,\lambda_\beta^C)\prod_{\lambda_\alpha^B}\prod_{\lambda_\beta^B}f(\lambda_\beta^B,\lambda_\alpha^B)$$
$$\times\prod_{\alpha=0}^{A}\mathscr{F}^{(a_\alpha,\theta_\alpha,\theta_{\alpha-1})}(\{\lambda_\alpha^C\},\{\lambda_\alpha^B\}). \qquad (4.12)$$

The sum is taken over the partitions of the sets $\{\lambda^B\}$ and $\{\lambda^C\}$ as in (4.10). Function $l_\gamma(\lambda)$ is defined as

$$l_\gamma(\lambda)=\prod_{\alpha=0}^{\gamma-1} l_{\alpha,\alpha+1}(\lambda).$$

Functions $\mathscr{F}^{(a,\theta,\theta')}$ are represented in terms of functions Z. For $a=0$:

$$\mathscr{F}^{(0,\theta,\theta')}(\{\lambda^C\}_N,\{\lambda^B\}_N)=\sum_{\text{part}}\prod_{\lambda^{DC}}\prod_{\lambda^{AC}} f(\lambda^{DC},\lambda^{AC}) \times\prod_{\lambda^{DB}}\prod_{\lambda^{AB}} f(\lambda^{DB},\lambda^{AB})Z_n(\{\lambda^{DB}\}_n,\{\lambda^{AC}\}_n) \times Z_{N-n}(\{\lambda^{DC}\}_{N-n},\{\lambda^{AB}\}_{N-n})\exp(\theta N_{AB}+\theta' N_{AC}). \tag{4.13}$$

The sum here is over all the partitions of the set $\{\lambda^B\}_N\cup\{\lambda^C\}_N$ into sets $\{\lambda^A\}_N$ and $\{\lambda^D\}_N$: $\{\lambda^B\}_N\cup\{\lambda^C\}_N=\{\lambda^A\}_N\cup\{\lambda^D\}_N$. The number of elements in each of these four sets is equal to N. For $a=\mp 1$:

$$\mathscr{F}^{(-1,\theta,\theta')}(\{\lambda^C\}_{N-1},\{\lambda^B\}_N)=-\mathrm{i}c^{1/2}\sum_{\text{part}}\prod_{\lambda^{AD}} f(\lambda^0,\lambda^{AD}) \times\prod_{\lambda^{DB}} f(\lambda^{DB},\lambda^0)\prod_{\lambda^{DC}}\prod_{\lambda^{AC}} f(\lambda^{DC},\lambda^{AC})\prod_{\lambda^{DB}}\prod_{\lambda^{AB}} f(\lambda^{DB},\lambda^{AB}) \times Z_n(\{\lambda^{DC}\}_n,\{\lambda^{AB}\}_n)Z_{N-n-1}(\{\lambda^{DB}\}_{N-n-1},\{\lambda^{AC}\}_{N-n-1}) \times\exp(\theta N_{AB}+\theta' N_{AC}) \tag{4.14}$$

$$\mathscr{F}^{(1,\theta,\theta')}(\{\lambda^C\}_N,\{\lambda^B\}_{N-1})=\mathrm{i}c^{1/2}\sum_{\text{part}}\prod_{\lambda^{AC}} f(\lambda^{AC},\lambda^0) \times\prod_{\lambda^{DC}} f(\lambda^0,\lambda^{DC})\prod_{\lambda^{DC}}\prod_{\lambda^{AC}} f(\lambda^{DC},\lambda^{AC})\prod_{\lambda^{DB}}\prod_{\lambda^{AB}} f(\lambda^{DB},\lambda^{AB}) \times Z_n(\{\lambda^{DC}\}_n,\{\lambda^{AB}\}_n)Z_{N-n-1}(\{\lambda^{DB}\}_{N-n-1},\{\lambda^{AC}\}_{N-n-1}) \times\exp\{\theta N_{AB}+\theta' N_{AC}\}. \tag{4.15}$$

The sum in (4.14) and (4.15) is taken over all the partitions

$$\{\lambda^B\}_N\cup\{\lambda^C\}_{N-1}=\{\lambda^A\}_{N-1}\cup\{\lambda^D\}_{N-1}\cup\{\lambda^0\}_1 \qquad \mathrm{card}\{\lambda^0\}_1=1$$

$$\{\lambda^B\}_{N-1}\cup\{\lambda^C\}_N=\{\lambda^A\}_{N-1}\cup\{\lambda^D\}_{N-1}\cup\{\lambda^0\}_1.$$

The scheme of the proof of representation (4.12) is given in appendix 3.

Now we briefly discuss the structure of representation (4.12). It represents the multi-field form factor (4.1) in terms of functions $\mathscr{F}^{(a,\theta,\theta')}$ which are, so to say, 'elementary form factors'. Indeed, function $\mathscr{F}^{(0,\theta,\theta')}$ is just the function $\sigma^{\theta-\theta'}$ of [4] which defines the form factor of the current. Functions $\mathscr{F}^{(\pm1,\theta,\theta')}$ are just single-field form factors of $\psi^+(0)$ and $\psi(0)$ investigated in detail in § 3 of this paper. Functions $\mathscr{F}^{(a,\theta,\theta')}$ satisfy the following relations at $\lambda_N^B\to\lambda_N^C$:

$$\mathscr{F}^{(a,\theta,\theta')}(\{\lambda^C\}_{N_C},\{\lambda^B\}_{N_B})\to\frac{\mathrm{i}c}{(\lambda_N^B-\lambda_N^C)} \times\left(\exp\theta\prod_{j=1}^{N_C-1} f(\lambda_N^C,\lambda_j^C)\prod_{j=1}^{N_B-1} f(\lambda_j^B,\lambda_N^B)-\exp(\theta')\prod_{j=1}^{N_C-1} f(\lambda_j^C,\lambda_N^C)\times\prod_{j=1}^{N_B-1} f(\lambda_N^B,\lambda_j^B)\right)\mathscr{F}^{(a,\theta,\theta')}(\{\lambda^C\}_{N_C-1},\{\lambda^B\}_{N_B-1}) +\text{regular terms}. \tag{4.16}$$

They are symmetric in the variables $\lambda_j^C \in \{\lambda^C\}_{N_C}$ and in the variables $\lambda_j^B \in \{\lambda^B\}_{N_B}$, decreasing as $O(1/\lambda)$ at $\lambda \to \infty$, $\lambda \in \{\lambda^B\} \cup \{\lambda^C\}$. These properties are sufficient to calculate the functions recursively by means of (4.16).

Form factors are thus described for the generalised model. The form factors in the NS model (1.1) are the particular case of those corresponding to choosing the parameters of the generalised model as $l_{\alpha,\alpha+1}(\lambda) = \exp[-i\lambda(x_{\alpha+1} - x_\alpha)]$ (cf (1.20)).

In the rest of the paper correlators are considered.

5. Mean values in the generalised model

Our aim is to study equal-time correlators in the NS model (1.1). These are defined as the normalised mean values of corresponding field operator products with respect to the physical ground state $|\Omega\rangle$ of the model. This quantity will be obtained in two steps. In this section the mean values of operator products with respect to Bethe eigenvectors (1.11) are studied in the generalised multi-point model. Then one passes over to the NS model by giving the corresponding values to the parameters of the generalised model (e.g. (1.8)) and taking the thermodynamical limit of the mean value. This is done in the next section. The results of §§ 5 and 6 are the direct generalisations of those obtained in [4, 5] for the simplest two-current correlator. The necessary proofs can be readily restored along the lines of those papers. So only brief discussion and final answers are given here.

We begin by studying properties of matrix elements of the corresponding operator products in the generalised multi-site model described in the previous section. Matrix elements $\mathcal{M}$ are given by the formula analogous to (4.1); the difference is that sets $\{\lambda^C\}_{N_C}$ and $\{\lambda^B\}_{N_B}$ are quite arbitrary and are not supposed to satisfy the systems of Bethe equations:

$$\mathcal{M}_{N_C N_B}(\{\lambda^C\}_{N_C}, \{\lambda^B\}_{N_B}) = \left\langle 0 \left| \prod_{j=1}^{N_C} \mathbb{C}(\lambda_j^C) \varphi_{a_0}(x_0) \mu_{\theta_0}(x_0, x_1) \right.\right.$$
$$\left.\left. \times \varphi_{a_1}(x_1) \mu_{\theta_1}(x_1, x_2) \ldots \varphi_{a_{A-1}}(x_{A-1}) \mu_{\theta_{A-1}}(x_{A-1}, x_A) \prod_{j=1}^{N_B} \mathbb{B}(\lambda_j^B) \right| 0 \right\rangle. \qquad (5.1)$$

Of primary importance is the analysis of singularities of this matrix element at $\lambda_j^B \to \lambda_k^C$. The structure of singularities is given in appendix 3. The main property is the existence of the first-order poles at $\lambda_j^C \to \lambda_k^B$ for $j = 1, \ldots, N_C$ and $k = 1, \ldots, N_B$, the residues at the poles being expressed in terms of the 'modified' matrix element:

$$\mathcal{M}_{N_C N_B}(\{\lambda^C\}_{N_C}, \{\lambda^B\}_{N_B})$$
$$\xrightarrow[\lambda_1^B \to \lambda_1^C]{} ic(\lambda_1^B - \lambda_1^C)^{-1} \sum_{\gamma=0}^{A} (l_{\gamma,\gamma+1}(\lambda_1^B) - l_{\gamma,\gamma+1}(\lambda_1^C)) \exp(\theta_\gamma) \prod_{\alpha<\gamma} l_{\alpha,\alpha+1}(\lambda_1^C)$$
$$\times \prod_{\beta>\gamma} l_{\beta,\beta+1}(\lambda_1^B) \prod_{j\neq 1}^{N_C} f(\lambda_1^C, \lambda_j^C) \prod_{j\neq 1}^{N_B} f(\lambda_1^B, \lambda_j^B)$$
$$\times \mathcal{M}^{\mathrm{mod}(\lambda_1,\gamma)}_{N_C-1,N_B-1}(\{\lambda^C\}_{N_C-1}, \{\lambda^B\}_{N_B-1}) + \ldots . \qquad (5.2)$$

All the notations used are the same as in appendix 3.

To obtain the mean value with respect to Bethe eigenvectors one has to put $N_B = N_C = N$, $\lambda_j^B = \lambda_j^C = \lambda_j, j = 1, \ldots, N$ and then impose the Bethe system (1.12) on

λ_j. It should be noted that in this case $A_+ = A_-$ also, $A_\pm$ denoting the number of $\varphi_{\pm 1}$ in (5.1). Functions $l_{\alpha,\alpha+1}(\lambda)$ are supposed to be smooth. For mean values the notation $\mathscr{D}_N$ will be used:

$$\mathscr{D}_N(\{\lambda\}_N, \{l_{\alpha,\alpha+1}\}, \{x_{\alpha,\alpha+1}\}) = \mathscr{M}_{N,N}(\{\lambda\}_N, \{\lambda\}_N). \tag{5.3}$$

It follows from (5.2) that $\mathscr{D}_N$ is a linear function of variables $x_{\gamma,\gamma+1}(\lambda_j)$

$$x_{\gamma,\gamma+1}(\lambda) \equiv \mathrm{i}\, \partial \log l_{\gamma,\gamma+1}(\lambda)/\partial\lambda.$$

The coefficient at $x_{\gamma,\gamma+1}(\lambda_j)$ can also be calculated from (5.2); it proves to be proportional to $\mathscr{D}_{N-1}$

$$\begin{aligned}(\partial/\partial x_{\gamma,\gamma+1}(\lambda_j))&\mathscr{D}_N(\{\lambda\}_N, \{l_{\alpha,\alpha+1}\}_{N,A-1}, \{x_{\alpha,\alpha+1}\}_{N,A})\\ &= c\left(\prod_{k\neq j} f(\lambda_j, \lambda_k) f(\lambda_k, \lambda_j)\right)\\ &\quad\times \mathscr{D}_{N-1}(\{\lambda\}_{N-1}, \{l^{\mathrm{mod}(\lambda_j,\gamma)}_{\alpha,\alpha+1}\}_{N-1,A-1}, \{x^{\mathrm{mod}(\lambda_j,\gamma)}_{\alpha,\alpha+1}\}_{N-1,A}).\end{aligned} \tag{5.4}$$

Here

$$\begin{aligned} &l^{\mathrm{mod}(\lambda,\gamma)}_{\alpha,\alpha+1}(\mu) = l_{\alpha,\alpha+1}(\mu) \qquad (\alpha \neq \gamma)\\ &l^{\mathrm{mod}(\lambda,\gamma)}_{\gamma,\gamma+1}(\mu) = l_{\gamma,\gamma+1}(\mu) f(\mu, \lambda)/f(\lambda, \mu)\\ &x^{\mathrm{mod}(\lambda,\gamma)}_{\alpha,\alpha+1}(\mu) = x_{\alpha,\alpha+1}(\mu) + \delta_{\alpha\gamma} K(\lambda, \mu)\\ &K(\lambda, \mu) = \mathrm{i}(\partial/\partial\lambda)\log(f(\lambda, \mu)/f(\mu, \lambda)) = 2c[(\lambda-\mu)^2 + c^2]^{-1}.\end{aligned}$$

The system (5.4) would define quantities $\mathscr{D}_N$ uniquely if the values of $\mathscr{D}_N$ at $x_{\gamma,\gamma+1}(\lambda_j) = 0$ were given. So let us define the irreducible part I_N of the mean value $\mathscr{D}_N$

$$I_N(\{\lambda\}_N, \{l_{\alpha,\alpha+1}\}_{N,A-1}) \equiv \mathscr{D}_N(\{\lambda\}_N, \{l_{\alpha,\alpha+1}\}_{N,A-1}, \{0\}). \tag{5.5}$$

It is readily proved that the irreducible part is expressed in terms of the form factor (4.1) as follows:

$$\begin{aligned} I_N(\{\lambda\}_N, &\{l_{\alpha,\alpha+1}\}_{N,A-1})\\ &= \lim_{\varepsilon\to 0} \mathscr{F}_N(\{\lambda_j\}_N, \{\lambda_j+\varepsilon\}_N)\big|_{l_{\alpha,\alpha+1}(\lambda^C_j) = l_{\alpha,\alpha+1}(\lambda^B_j) = l_{\alpha,\alpha+1}(\lambda_j)}.\end{aligned} \tag{5.6}$$

The representation (4.13)–(4.15) for the form factor results in the following structure of the irreducible part:

$$I_N(\{\lambda\}_N, \{l_{\alpha,\alpha+1}\}_{N,A-1}) = \sum_{\mathrm{part}} \mathscr{A}_N(\{\lambda^+\}, \{\lambda^-\}, \{\lambda^0\}) \prod_{\alpha=1}^{A} \prod_{\lambda^+_\alpha} l_\alpha(\lambda^+_\alpha) \prod_{\lambda^-_\alpha} l^{-1}_\alpha(\lambda^-_\alpha). \tag{5.7}$$

The sum here is taken over the partitions

$$\{\lambda\}_N = \left(\bigcup_{\alpha=1}^{A} \{\lambda^+_\alpha\} \cup \{\lambda^-_\alpha\}\right) \cup \{\lambda^0\}$$

$$\mathrm{card}\{\lambda^+_\alpha\} = \mathrm{card}\{\lambda^-_\alpha\}.$$

Functions $\mathscr{A}_N$ are 'Fourier coefficients' of the irreducible part; they do not depend on the parameters $l_{\gamma,\gamma+1}(\lambda)$ of the generalised model and are defined by the R matrix only.

To construct the solution for the system (5.4) with complementary conditions (5.5) one introduces functions $\mathscr{E}_N$

$$\mathscr{E}_N(\{\lambda^+\},\{\lambda^-\},\{\lambda\}_N,\{x\})$$

$$=\sum_{\text{part}}\prod_{\gamma=0}^{A}\det(\varphi^{jk}_{\gamma,\gamma+1}(\{\lambda_\gamma\}))\exp\left(\sum_\gamma \theta_\gamma n_\gamma\right)$$

$$\times\prod_{\alpha=0}^{A-1}\prod_{\alpha\geqslant\gamma}\prod_{\lambda^+_\alpha}(f(\lambda^+_\alpha,\lambda_\gamma)/f(\lambda_\gamma,\lambda^+_\alpha))\prod_{\lambda^-_\alpha}(f(\lambda_\gamma,\lambda^-_\alpha)/f(\lambda^-_\alpha,\lambda_\gamma)). \tag{5.8}$$

The sum here is over the partitions $\{\lambda\}_N=\bigcup_{\gamma=0}^{A}\{\lambda_\gamma\}$

$$\varphi^{jk}_{\gamma,\gamma+1}(\{\lambda\})=\left(x_{\gamma,\gamma+1}(\lambda_j)+\sum_l K(\lambda_j,\lambda_l)\right)\delta_{jk}-K(\lambda_j,\lambda_k).$$

Functions $\mathscr{E}_N$ are defined uniquely by the following system of equations:

$$\partial\mathscr{E}_N/\partial x_{\gamma,\gamma+1}(\lambda_j)=\exp\{\theta_\gamma\}\prod_{\alpha\geqslant\gamma}\prod_{\lambda^+_\alpha}(f(\lambda^+_\alpha,\lambda_j)/f(\lambda_j,\lambda^+_\alpha))$$

$$\times\prod_{\lambda^-_\alpha}(f(\lambda_j,\lambda^-_\alpha)/f(\lambda^-_\alpha,\lambda_j))\mathscr{E}_{N-1}(\{\lambda^+\},\{\lambda^-\},\{\lambda\}_{N-1},\{x^{\operatorname{mod}(\lambda_j,\gamma)}_{\alpha,\alpha+1}\}) \tag{5.9}$$

and by the complementary condition

$$\mathscr{E}_N(\{\lambda^+\},\{\lambda^-\},\{\lambda\}_N,\{0\})=\delta_{N,0}. \tag{5.10}$$

Parameters x^{mod} in (5.9) are defined in the same way as in (5.4). The solution of the system (5.4) with the complementary condition (5.7) is now given as

$$\mathscr{D}_N=c^N\prod_{j\neq k}f(\lambda_j,\lambda_k)\prod_{j=1}^{N}r(\lambda_j)\sum_{\text{part}_1}\sum_{\text{part}_2}\prod_{\alpha=1}^{A}\prod_{\lambda^+_\alpha}l_\alpha(\lambda^+_\alpha)$$

$$\times\prod_{\lambda^-_\alpha}l^{-1}(\lambda^-_\alpha)A_k(\{\lambda^+\},\{\lambda^-\},\{\lambda^0\})\mathscr{E}_{N-k}(\{\lambda^+\},\{\lambda^-\},\{\lambda^{v}\}_{N-k},\{x\}). \tag{5.11}$$

The first sum here is over the partitions

$$\{\lambda\}_N=\{\lambda^{I}\}_k\cup\{\lambda^{v}\}_{N-k}$$

and the second sum is over the partitions

$$\{\lambda^{I}\}_k=\left(\bigcup_{\alpha=1}^{A}(\{\lambda^+_\alpha\}\cup\{\lambda^-_\alpha\})\right)\cup\{\lambda^0\}.$$

Formula (5.11) is readily established by direct calculations. For the normalised mean value one obtains (taking into account (1.14)):

$$\mathbb{D}_N=\left\langle\Omega\left|\prod_{\alpha=0}^{A}\varphi_{a_\alpha}(x_\alpha)\mu_{\theta_\alpha}(x_\alpha,x_{\alpha+1})\right|\Omega\right\rangle(\langle\Omega|\Omega\rangle)^{-1}$$

$$=\sum_{\text{part}_1}\sum_{\text{part}_2}\mathscr{A}_k(\{\lambda^+\},\{\lambda^-\},\{\lambda^0\})E_{N-k}(\{\lambda^+\},\{\lambda^-\},\{\lambda^{v}\})$$

$$\times(\det(\varphi^{v}_{ij})/\det(\varphi_{ij}))\prod_{\alpha=1}^{A}\prod_{\lambda^+_\alpha}l_\alpha(\lambda^+_\alpha)\prod_{\lambda^-_\alpha}l^{-1}_\alpha(\lambda^-_\alpha). \tag{5.12}$$

The sum here is taken as in (5.11). Function E_N differs from function $\mathscr{E}_N$ only by the normalising factor

$$E_N(\{\lambda^+\},\{\lambda^-\},\{\lambda\}_N)=\frac{\mathscr{E}_N(\{\lambda^+\},\{\lambda^-\},\{\lambda\}_N,\{x\})}{\det(\varphi_{ij})}. \tag{5.13}$$

Matrices $\varphi_{ij} = \varphi(\lambda_i, \lambda_j)$ and $\varphi^v_{ij} = \varphi(\lambda^v_i, \lambda^v_j)$ are obtained by differentiation of functions φ_j in λ as $\varphi_{ij} = \partial\varphi(\lambda_j)/\partial\lambda_i$

$$\varphi(\lambda_i, \lambda_j) = \left(\rho_0(\lambda_j) + \sum_{l=1}^{N} K(\lambda_i, \lambda_l)\right)\delta_{ij} - K(\lambda_i, \lambda_j)$$
$$\rho_0(\lambda) = \mathrm{i}\,\partial(\log r(\lambda))/\partial\lambda. \tag{5.14}$$

The representation (5.12) is the main result of the QISM analysis of operator product mean values in the generalised model. It should be noted that at $A_\pm = 0$, $A_0 = 2$ one has just the two-current mean value investigated in detail in [4].

6. Many-point correlators in the NS model

Turn now to the NS model (1.1). Mean values of operator products in this model are given by the general formula (5.12) if one takes functions $l_{\alpha,\alpha+1}(\lambda)$ and $x_{\alpha,\alpha+1}(\lambda)$ in accordance with the structure of the NS monodromy matrix (1.16) and (1.17) (to be compared with (1.20))

$$l_{\alpha,\alpha+1}(\lambda) = \exp[-\mathrm{i}(x_{\alpha+1} - x_\alpha)\lambda]$$
$$x_{\alpha,\alpha+1}(\lambda) = x_{\alpha+1} - x_\alpha \qquad x_0 = 0 \qquad x_{A+1} = L$$

We further consider the model in the thermodynamical limit which is of primary interest. In this limit the length of the box L and the number N of spectral parameters in Bethe eigenvectors in (5.12) become infinitely large, the ratio N/L remaining finite:

$$N \to \infty \qquad L \to \infty \qquad \rho = N/L < \infty.$$

Correlators are the normalised mean values with respect to the ground state $|\Omega\rangle$. The corresponding spectral parameters fill the Fermi zone $-q \leqslant \lambda \leqslant q$, the density $\rho(\lambda_j) = (\lambda_{j+1} - \lambda_j)^{-1}L^{-1}$ satisfying the linear integral equation [8]:

$$\rho(\lambda) - (1/2\pi)\int_{-q}^{q} K(\lambda, \mu)\rho(\mu)\,\mathrm{d}\mu = (1/2\pi). \tag{6.1}$$

The parameter q is just the Fermi bare momentum; the function $\rho_0(\lambda)$ in (5.14) is now $\rho_0(\lambda) \equiv 1$. The density of the particles N/L in the coordinate space is equal to the integral over the Fermi zone:

$$N/L = \int_{-q}^{q} \rho(\lambda)\,\mathrm{d}\lambda. \tag{6.2}$$

To obtain the correlator one has to take the thermodynamical limit in (5.12). It is done quite similar to the case of the two-current correlator, so only the result and main points of the proof are given below. The correlator is represented in the following form:

$$\mathbb{D}_\infty = \left\langle \Omega \middle| \prod_{\alpha=0}^{A} \varphi_{a_\alpha}(x_\alpha)\mu_{\theta_\alpha}(x_\alpha, x_{\alpha+1}) \middle| \Omega \right\rangle (\langle\Omega|\Omega\rangle)^{-1}$$
$$= \sum_{k\geqslant 0} \frac{1}{k!} \int_{-q}^{q} \left(\prod_{j=1}^{k} \frac{\omega(\lambda_j)\,\mathrm{d}\lambda_j}{2\pi}\right) \sum_{\{\lambda\}_k = (\bigcup_{\alpha=1}^{A}\{\lambda_\alpha^+\}\cup\{\lambda_\alpha^-\})\cup\{\lambda^0\}} \mathscr{A}_k(\{\lambda^+\}, \{\lambda^-\}, \{\lambda^0\})$$
$$\times \prod_{\alpha=1}^{A} \prod_{\lambda_\alpha^+} l_\alpha(\lambda_\alpha^+) \prod_{\lambda_\alpha^-} l_\alpha^{-1}(\lambda_\alpha^-) \mathbb{E}_k(\{\theta_\alpha\}, \{x_{\alpha-1,\alpha}\}, \{\lambda^+\}, \{\lambda^-\}). \tag{6.3}$$

Here the weight $\omega(\lambda)$ is

$$\omega(\lambda)=\exp\left(-(1/2\pi)\int_{-q}^{q}K(\lambda,\mu)\,\mathrm{d}\mu\right) \tag{6.4}$$

and the $\mathscr{A}_k$ are the Fourier coefficients of the irreducible part I_k defined in (5.7). The function $\mathbb{E}_k$ is the limit of E_{N-k} at $N\to\infty$; this function depends on parameters θ_α, on spectral parameters $\{\lambda_\alpha^\pm\}$ and on functions $x_{\alpha,\alpha+1}(\lambda)$ (in the NS model $x_{\alpha,\alpha+1}(\lambda)=x_{\alpha+1}-x_\alpha$). Function $\mathbb{E}_k$, $k\geqslant 1$, is expressed in terms of functions P^α which are solutions of non-linear integral equations

$$\mathbb{E}_k(\{\theta_\alpha\},\{x_{\alpha-1,\alpha}\},\{\lambda^+\},\{\lambda^-\})=\prod_{\alpha=1}^{A}\exp\left(\int_{-q}^{q}x_{\alpha-1,\alpha}(t)P^\alpha(t,\{\lambda^+\},\{\lambda^-\})\,\mathrm{d}t\right) \tag{6.5}$$

$$2\pi P^\gamma(t)=\prod_{\alpha\geqslant\gamma}\prod_{\lambda_\alpha^+}(f(\lambda_\alpha^+,t)/f(t,\lambda_\alpha^+))\prod_{\lambda_\alpha^-}(f(t,\lambda_\alpha^-)/f(\lambda_\alpha^-,t))$$
$$\times\exp\left(\theta_\gamma+\int_{-q}^{q}K(t,s)P^\gamma(s)\,\mathrm{d}s\right)-1. \tag{6.6}$$

The solution of (6.6) can be proved to exist and to be unique.

Let us now make some comments on how the representation (6.3) is obtained. All the dependence on spectral parameters $\{\lambda^v\}$ in (5.12) enters the ratio of the determinants and functions E_{N-k}. The thermodynamical limit of the determinants' ratio gives the weight $\omega(\lambda)$ (6.4). Summing over the partitions $\{\lambda\}_N=\{\lambda^I\}_k\cup\{\lambda^v\}_{N-k}$ results in integration over spectral parameters $\{\lambda^I\}_k$. The limit at $N\to\infty$ of function E_{N-k} is obtained as follows. First one takes the limit not of the function itself but of the equation for this function, obtaining the following system of equations in variational derivatives:

$$\delta\mathbb{E}_k(\{\theta_\alpha\},\{x_{\alpha-1,\alpha}\},\{\lambda^+\},\{\lambda^-\})/\delta x_{\gamma,\gamma+1}(\lambda)=-(1/2\pi)$$
$$\times\mathbb{E}_k(\{\theta_\alpha\},\{x_{\alpha-1,\alpha}\},\{\lambda^+\},\{\lambda^-\})+\exp(\theta_\gamma)\prod_{\alpha\geqslant\gamma}\prod_{\lambda_\alpha^+}\frac{f(\lambda_\alpha^+,\lambda)}{f(\lambda,\lambda_\alpha^+)}$$
$$\times\prod_{\lambda_\alpha^-}\frac{f(\lambda,\lambda_\alpha^-)}{f(\lambda_\alpha^-,\lambda)}\mathbb{E}_k(\{\theta_\alpha\},\{x_{\alpha-1,\alpha}(\mu)+\delta_{\alpha\gamma}K(\lambda,\mu)\},\{\lambda^+\},\{\lambda^-\}). \tag{6.7}$$

The normalisation (5.10) means that

$$\mathbb{E}_k(\{\theta_\alpha\},\{0\},\{\lambda^+\},\{\lambda^-\}=1. \tag{6.8}$$

Function $\mathbb{E}_k$ (6.5) and (6.6) is easily seen to be the solution of (6.7) and (6.8).

The many-point correlation functions of operators $\psi(x)$, $\psi^+(x)$ and $\mu_\theta(x,y)$ are thus obtained. The non-local operator $\mu_\theta(x,y)$ (4.3) can be interpreted as a 'disorder' operator [13]. From the point of view of the Bose gas model, however, the local current operator $j(x)=\psi^+(x)\psi(x)$ seems to be more natural. This can be obtained by differentiating $\mu_\theta(x,y)$ at $\theta=0$. For example, the following expression is valid for the three-current correlator:

$$\langle j(0)j(x_1)j(x_2)\rangle=-\tfrac{1}{2}(\partial^3/\partial x_1^2\partial x_2)\left\langle\int_0^{x_1}j(z)\,\mathrm{d}z\int_{x_1}^{x_2}j(y)\,\mathrm{d}y\right\rangle$$
$$=-\tfrac{1}{4}(\partial^3/\partial x_1^2\partial x_2)(\partial^3/\partial\theta_0^2\partial\theta_1)\langle\mu_{\theta_0}(0,x_1)\mu_{\theta_1}(x_1,x_2)\rangle|_{\theta_0=\theta_1=0}. \tag{6.9}$$

7. Two-field correlator

In this section the two-field correlator is considered in more detail. Operators $\psi(x)\psi(y)$ and $\psi^+(x)\psi^+(y)$ do not conserve the number of particles and the corresponding correlators are equal to zero. The correlator $\psi^+(x)\psi(y)$ depends on $x-y$ only due to the translation invariance, so one can put $y=0$. Using the results of the previous section, one has this correlator:

$$\mathbb{D}(x)=\langle\Omega|\psi^+(x)\psi(0)|\Omega\rangle(\langle\Omega|\Omega\rangle)^{-1}=\sum_{k\geq 1}\frac{1}{k!}\int_{-q}^{q}\prod_{j=1}^{k}\left(\frac{\omega(\lambda_j)\,\mathrm{d}\lambda_j}{2\pi}\right)I_k^d(\{\lambda\}) \tag{7.1}$$

where the weight $\omega(\lambda)$ is given in (6.4) and the 'dressed' irreducible part I_k^d is defined as

$$I_k^d(\{\lambda\})=\sum_{\{\lambda\}_k=\{\lambda^+\}_{n+1}\cup\{\lambda^-\}_n\cup\{\lambda^0\}_{k-2n-1}}\mathscr{A}_k(\{\lambda^+\},\{\lambda^-\},\{\lambda^0\})$$
$$\times\exp\left[-\mathrm{i}x\left(\sum_{j=1}^{n+1}\lambda_j^+-\sum_{j=1}^{n}\lambda_j^-\right)+x\int_{-q}^{q}P_n(t,\{\lambda^+\},\{\lambda^-\})\,\mathrm{d}t\right]. \tag{7.2}$$

Function $P_n(t)$ here is the solution of the following non-linear integral equation:

$$2\pi P_n(t)=\prod_{j=1}^{n+1}(f(\lambda_j^+,t)/f(t,\lambda_j^+))\prod_{j=1}^{n}(f(t,\lambda_j^-)/f(\lambda_j^-,t))$$
$$\times\exp\left(\int_{-q}^{q}K(t,s)P_n(s)\,\mathrm{d}s\right)-1. \tag{7.3}$$

Functions $\mathscr{A}_k(\{\lambda^+\},\{\lambda^-\},\{\lambda^0\})$ are Fourier coefficients of the irreducible part of the two-field form factor. As explained in § 4 (see (4.12)), this form factor is expressed in terms of one-field form factors described in § 3

$$\mathscr{F}_N(\{\lambda^C\}_N,\{\lambda^B\}_N)=\left\langle 0\left|\prod_{j=1}^{N}\mathbb{C}(\lambda_j^C)\psi^+(x)\psi(0)\prod_{j=1}^{N}\mathbb{B}(\lambda_j^B)\right|0\right\rangle$$
$$=\sum_{\text{part}}\prod_{\lambda_1^C}l(\lambda_1^C)\prod_{\lambda_1^B}l^{-1}(\lambda_1^B)\prod_{\lambda_1^B,\lambda_2^B}f(\lambda_2^B,\lambda_1^B)\prod_{\lambda_1^C,\lambda_2^C}f(\lambda_1^C,\lambda_2^C)$$
$$\times\langle\{\lambda_1^C\}|\psi^+(0)|\{\lambda_1^B\}\rangle\langle\{\lambda_2^B\}|\psi(0)|\{\lambda_2^C\}\rangle. \tag{7.4}$$

The sum here is taken over the following partitions of sets $\{\lambda^C\}$, $\{\lambda^B\}$:

$$\{\lambda^C\}_N=\{\lambda_1^C\}_{n+1}\cup\{\lambda_2^C\}_{N-n-1}\qquad\{\lambda^B\}_N=\{\lambda_1^B\}_n\cup\{\lambda_2^B\}_{N-n}.$$

Irreducible parts can be expressed by means of (5.6) in terms of this form factor. Straightforward (though somewhat tiresome) calculations give the following results for the first three irreducible parts:

$$I_1(\lambda_1)=cl(\lambda_1)$$
$$I_2(\lambda_1,\lambda_2)=-4g^2(\lambda_1,\lambda_2)f(\lambda_1,\lambda_2)l(\lambda_1)-4g^2(\lambda_2,\lambda_1)f(\lambda_2,\lambda_1)l(\lambda_2)$$
$$I_3(\lambda_1,\lambda_2,\lambda_3)=(4/c^2)f(\lambda_2,\lambda_1)f(\lambda_3,\lambda_1)g^2(\lambda_2,\lambda_1)g^2(\lambda_3,\lambda_1)$$
$$\times(f(\lambda_1,\lambda_2)f(\lambda_1,\lambda_3)f(\lambda_3,\lambda_2)f(\lambda_2,\lambda_3))^{-1}l(\lambda_2)l(\lambda_3)l^{-1}(\lambda_1)$$
$$-g(\lambda_1,\lambda_2)g(\lambda_1,\lambda_3)(f(\lambda_2,\lambda_1)f(\lambda_3,\lambda_1))^{-1}[2\lambda_{12}^{-1}\lambda_{13}^{-1}-\lambda_{23}^{-1}-(\lambda_{23}^2+c^2)^{-1}$$
$$\times(\lambda_{13}\lambda_{12}^{-1}+\lambda_{12}\lambda_{13}^{-1}-1)]$$
$$+(\text{cyclic permutations of }1,2,3)\qquad\lambda_{ij}=\lambda_i-\lambda_j. \tag{7.5}$$

So one obtains for the first three terms of the series (7.1)

$$\langle\psi^+(x)\psi(0)\rangle=\int_{-q}^{q} I_1^d(\lambda)\frac{\omega(\lambda)\,\mathrm{d}\lambda}{2\pi}+\frac{1}{2}\int_{-q}^{q} I_2(\lambda_1,\lambda_2)\frac{\omega(\lambda_1)\omega(\lambda_2)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2}{(2\pi)^2}$$
$$+\frac{1}{3!}\int_{-q}^{q} I_3^d(\lambda_1,\lambda_2,\lambda_3)\frac{\omega(\lambda_1)\omega(\lambda_2)\omega(\lambda_3)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2\,\mathrm{d}\lambda_3}{(2\pi)^3}+\ldots. \tag{7.6}$$

This expansion is quite similar from the formal point of view to the corresponding expansion for the two-current correlator obtained in [5]. However, the essential difference is in the properties at the strong coupling limit ($c\to\infty$). Only the first two terms of the series similar to (7.6) for the current correlator survive at $c\to\infty$. For the field correlator all the terms of the expansion (7.6) remain finite at $c=\infty$. Let us demonstrate that it is indeed the case and write down the series at this limit. One has that $\omega(\lambda)\to 1$, $P_n(t)\to -1/\pi$ and thus the field correlator at $c=\infty$ is

$$\mathbb{D}(x)=\exp(-xq/\pi)\sum_{k\geqslant 1}\frac{1}{k!}\int_{-q}^{q}\left(\prod_{j=1}^{k}\frac{\mathrm{d}\lambda_j}{2\pi}\right)I_k(\{\lambda\}). \tag{7.7}$$

Here $I_k(\{\lambda\})$ can be obtained from the corresponding form factor at $c\to\infty$ as

$$I_k(\{\lambda\})=\lim_{\varepsilon\to 0}\mathbb{F}_k(\{\lambda_j\},\{\lambda_j+\varepsilon\},\{l\},\{l\})|_{l_j=\exp(-\mathrm{i}x\lambda_j)} \tag{7.8}$$

$$\mathbb{F}_k(\{\lambda^C\},\{\lambda^B\},\{l^C\},\{l^B\})=\lim_{c\to\infty}\left(c^k\prod_{j<i}f(\lambda_j^B,\lambda_i^B)f(\lambda_i^C,\lambda_j^C)\right)^{-1}$$
$$\times\mathscr{F}_k(\{\lambda^C\}_k,\{\lambda^B\}_k,\{l^C\}_k,\{l^B\}_k). \tag{7.9}$$

Taking the limit in (7.9) one obtains for $\mathbb{F}_k$:

$$\mathbb{F}_k(\{\lambda^C\},\{\lambda^B\},\{l^C\},\{l^B\})=\left(\prod_{i<j}\lambda_{ij}^B\lambda_{ji}^C\right)\sum_{\text{part}}\prod_{\lambda_{pr}^C}l(\lambda_{pr}^C)$$
$$\times\prod_{\lambda_{pr}^B}l^{-1}(\lambda_{pr}^B)\prod_{\lambda_{pr}^C,\lambda_{ab}^C}(\lambda_{pr}^C-\lambda_{ab}^C)^{-1}\prod_{\lambda_{pr}^B}(\lambda_{pr}^C-\lambda_{pr}^B)^{-1}$$
$$\times\prod_{\lambda_{ab}^B,\lambda_{pr}^B}(\lambda_{ab}^B-\lambda_{pr}^B)^{-1}\prod_{\lambda_{ab}^C}(\lambda_{ab}^C-\lambda_{ab}^B)^{-1}. \tag{7.10}$$

The sum is taken over partitions $\{\lambda^C\}_k=\{\lambda_{pr}^C\}_n\cup\{\lambda_{ab}^C\}_{K-n}$ and $\{\lambda^B\}_K=\{\lambda_{pr}^B\}_{n-1}\cup\{\lambda_{ab}^B\}_{K-n+1}$.

The expression for the correlator at $c\to\infty$ is thus obtained, and differs from the well known formulae of [14, 15]. Let us now explain the relation of (7.9) to the results of those papers.

It was shown in [16] that at $c\to\infty$ the field correlator in the Bose gas can be obtained by means of the Jordan–Wiegner transform. To do this one must use the following representation of the Bose field $\psi(x)$ in terms of the auxiliary Fermi field $\phi(x)$:

$$\psi(x)=\exp\left(\mathrm{i}\pi\int_x^{\infty}\phi^+(y)\phi(y)\,\mathrm{d}y\right)\phi(y). \tag{7.11}$$

The physical vacuum (ground state) $|\Omega\rangle$ is now defined by the relations

$$\phi(\lambda)=\int_{-\infty}^{\infty}\exp(\mathrm{i}\lambda x)\phi(x)\,\mathrm{d}x \tag{7.12}$$

$$\phi(\lambda)|\Omega\rangle=0\qquad |\lambda|>q\qquad \phi^+(\lambda)|\Omega\rangle=0\qquad |\lambda|<q\qquad \langle\Omega|\Omega\rangle=1. \tag{7.13}$$

The Bose field correlator is

$$\mathbb{D}(x)=\langle\Omega|\psi^+(x)\psi(0)|\Omega\rangle = \left\langle\Omega\left|\phi^+(x)\exp\left(\mathrm{i}\pi\int_0^x\phi^+(y)\phi(y)\,\mathrm{d}y\right)\phi(0)\right|\Omega\right\rangle. \tag{7.14}$$

The Lenard formula is obtained from this expression as follows. One does the ordering of the exponential with respect to the bare vacuum (putting all the operators ϕ^+ to the left of operators ϕ)

$$\exp\left(\mathrm{i}\pi\int_0^x\phi^+(y)\phi(y)\,\mathrm{d}y\right)=N_0\exp\left(-2\int_0^x\phi^+(y)\phi(y)\,\mathrm{d}y\right). \tag{7.15}$$

Here N_0 is the normal ordering symbol. Putting (7.15) into (7.14) and using (7.13) one obtains

$$\mathbb{D}(x)=\sum_{k\geqslant0}\left(\frac{(-2)^k}{k!}\right)\int_0^x\mathrm{d}y_1\ldots\mathrm{d}y_k\langle\Omega|\phi_-^+(x)\phi_-^+(y_1)\ldots\phi_-^+(y_k)\times\phi_-(y_k)\ldots\phi_-(y_1)\phi_-(0)|\Omega\rangle \tag{7.16}$$

where

$$\phi_-(x)=\int_{-q}^{q}\exp(-\mathrm{i}\lambda x)\phi(\lambda)(\mathrm{d}\lambda/2\pi). \tag{7.17}$$

We next introduce the notation

$$\begin{aligned}&\{\phi_-^+(x),\phi_-(0)\}=\int_{-q}^{q}\exp(-\mathrm{i}\lambda x)(\mathrm{d}\lambda/2\pi)=\mathcal{K}(x)/\pi\\&\mathcal{K}(x)=x^{-1}\sin qx\qquad\mathcal{K}(x,y)\equiv\mathcal{K}(x-y).\end{aligned} \tag{7.18}$$

Now the vacuum mean value in (7.16) can be calculated using the Wick theorem

$$\langle\Omega|\phi_-^+(x)\phi_-^+(y_1)\ldots\phi_-^+(y_k)\phi_-(y_k)\ldots\phi_-(y_1)\phi_-(0)|\Omega\rangle = \det\begin{pmatrix}\mathcal{K}(x,0) & \mathcal{K}(x,y_1) & \ldots & \mathcal{K}(x,y_k)\\ \mathcal{K}(y_1,0) & \mathcal{K}(y_1,y_1) & \ldots & \mathcal{K}(y_1,y_k)\\ \vdots & \vdots & & \vdots\\ \mathcal{K}(y_k,0) & \mathcal{K}(y_k,y_1) & \ldots & \mathcal{K}(y_k,y_k)\end{pmatrix}\equiv d_k(x,\{y\}). \tag{7.19}$$

Putting this into (7.16) one arrives at the Lenard formula

$$\mathbb{D}(x)=\sum_{k\geqslant0}\frac{(-2/\pi)^k}{k!}\int_0^x\left(\prod_{j=1}^{k}\mathrm{d}y_j\right)d_k(x,\{y\}). \tag{7.20}$$

To obtain the representation (7) one takes the integrals over y_j and uses (7.18) for a function $\mathcal{K}(x,y)$. The terms containing $\mathcal{K}(y_i,y_i)$ in (7.20) must be considered separately, the summing up of these resulting in the exponential factor. Let us show how the first term of the series (7.7) is obtained from (7.20). To do this one extracts the term $\mathcal{K}(x,0)\mathcal{K}(y_1,y_1)\ldots\mathcal{K}(y_k,y_k)$ from all the functions $d_k(x,\{y\}_k)$. Summing up over k and integrating over y_i ($i=1,\ldots,k$) results in

$$\mathscr{D}^{(1)}(x)=\exp(-2qx/\pi)\mathcal{K}(x,0) \tag{7.21}$$

which is just the first term at the right-hand side of (7.7).

The extraction of the exponential factor is physically quite understandable and can be done directly in the expression (7.14). One represents the field ϕ as

$$\phi(x)=\phi_+(x)+\phi_-(x) \qquad \phi_+(x)=\int_{|\lambda|>q}\exp(-\mathrm{i}\lambda x)\phi(\lambda)\left(\frac{\mathrm{d}\lambda}{2\pi}\right) \tag{7.22}$$

(the operator $\phi_-(x)$ is defined in (7.17)). The normal ordering of an operator with respect to the physical vacuum $|\Omega\rangle$ means that all the operators ϕ_+^+ are put to the left of ϕ_+ and all the operators ϕ_- are put to the left of ϕ_-^+. This normal ordering is to be done by means of the following commutation relations:

$$\begin{aligned}&\{\phi_-(x),\phi_-^+(y)\}=\mathcal{K}(x,y)/\pi \qquad \{\phi_\pm(x),\phi_\mp^+(y)\}=0\\&\{\phi_+(x),\phi_+^+(y)\}=\delta(x-y)-\mathcal{K}(x,y)/\pi.\end{aligned} \tag{7.23}$$

Equation (7.15) can be rewritten as

$$\exp\left(\mathrm{i}\pi\int_0^x\phi^+(y)\phi(y)\,\mathrm{d}y\right)=N_+\tilde{N}_-\exp\left(-2\int_0^x\phi^+(y)\phi(y)\,\mathrm{d}y\right) \tag{7.24}$$

where N_+ means the normal ordering of fields ϕ_+ and $\tilde{N}_-$ is the antinormal ordering of fields ϕ_-. It is now easy to see that the extraction of terms containing $\mathcal{K}(y_i,y_i)$ in (7.20) (which are combined to give the exponential factor) is equivalent to the normal ordering of the current in (7.24)

$$\begin{aligned}&\exp\left(\mathrm{i}\pi\int_0^x\phi^+(y)\phi(y)\,\mathrm{d}y\right)\\&\qquad=\exp(-2qx/\pi)N_+\tilde{N}_-\exp\left(-2\int_0^x N_+N_-(\phi^+(y)\phi(y))\,\mathrm{d}y\right).\end{aligned} \tag{7.25}$$

Expanding the exponential (7.25) in the formula (7.14) and going to the Fourier transforms of operators $\phi(x)$ one comes to the expansion (6.7).

Appendix 1

Functions $Z_n(\{\lambda\}_n,\{\mu\}_n)$ were introduced and studied in detail in [11]. Here the definition of these functions is given as well as the recursive relations permitting the calculation of any of them.

Function $Z_n(\{\lambda\}_n,\{\mu\}_n)$ is equal to the partition function of a model on a square $n\times n$ lattice. The states $k=1,2$ are related to the links of the lattice. The Boltzmann weight at a site is determined by the states at the links adjacent to the site. If the horizontal links with states k_1, k_4 and the vertical links with states k_2, k_3 enter the site located at the ith row and the kth column of the lattice then the Boltzmann weight prescribed to the site is

$${}_{k_1k_2}W^{ij}_{k_3k_4}={}_{k_1k_2}R(\lambda_i,\mu_j)_{k_3k_4}$$

where the matrix $R(\lambda,\mu)$ is defined in (1.4)

$$\begin{aligned}&{}_{11}R(\lambda,\mu)_{11}={}_{22}R(\lambda,\mu)_{22}=f(\mu,\lambda)\\&{}_{12}R(\lambda,\mu)_{12}={}_{21}R(\lambda,\mu)_{21}=1\\&{}_{21}R(\lambda,\mu)_{12}={}_{12}R(\lambda,\mu)_{21}=g(\mu,\lambda).\end{aligned}$$

Boundary conditions for Z_n are defined so that $k_i=1$ for all the links going from the upper side and from the left side of the lattice and $k_i=2$ for all the links going from the lower side and from the right side. The definition of Z_n is thus given.

At $\lambda_n \to \mu_n$ the partition function Z_n is reduced to Z_{n-1} according to the rule

$$Z_n(\{\lambda\}_n,\{\mu\}_n) \to -\mathrm{i}c(\lambda_n-\mu_n)^{-1}\prod_{j=1}^{n-1} f(\mu_j,\mu_n)f(\lambda_n,\lambda_j) \times Z_{n-1}(\{\lambda\}_{n-1},\{\mu\}_{n-1})+\dots \tag{A1.1}$$

(the terms regular at $\lambda_n\to\mu_n$ are not included here). It is not difficult to establish that $Z_n(\{\lambda\}_n,\{\mu\}_n)=\mathrm{O}(1/\lambda_j)(\mathrm{O}(1/\mu_k))$ if $\lambda_j\to\infty$ $(\mu_k\to\infty)$ and all the other variables are fixed. So extracting all the singularities

$$Z_n(\{\lambda\}_n,\{\mu\}_n)=P_n(\{\lambda\}_n,\{\mu\}_n)\left(\prod_{j,k=1}^{n}(\lambda_j-\mu_k)\right)^{-1} \tag{A1.2}$$

one comes to the conclusion that the function P_n is a polynomial in each variable λ_j or μ_k. This permits one to recover P_n from P_{n-1} by means of (A1.1) using the Lagrange interpolation formula.

Appendix 2

Equations (3.4) and (3.5) result in the following relations for the polynomials $\mathscr{P}_N$:

$$\mathscr{P}_N(\{\lambda^C\}_{N-1},\{\lambda^B\}_N)|_{\lambda_1^C=\lambda_1^B}=\mathrm{i}c\left(\prod_{j=2}^{N-1}(\lambda_{1j}^C+\mathrm{i}c)\prod_{j=2}^{N}(\lambda_{j1}^B+\mathrm{i}c) - \prod_{j=2}^{N-1}(\lambda_{j1}^C+\mathrm{i}c)\prod_{j=2}^{N}(\lambda_{1j}^B+\mathrm{i}c)\right)\mathscr{P}_{N-1}(\{\lambda^C\}_{N-2},\{\lambda^B\}_{N-1}). \tag{A2.1}$$

Let us show that $\mathscr{P}_N$ is a polynomial of degree $N-2$ in any of λ_j^B, λ_k^C. The equivalent statement is that the function $\mathscr{F}_N(\{\lambda^C\}_{N-1},\{\lambda^B\}_N)$ decreases as $(\lambda_j^B)^{-1}$ at $\lambda_j^B\to\infty$ (and other variables fixed) and as $(\lambda_k^C)^{-2}$ at $\lambda_k^C\to\infty$. The form factor $\mathscr{F}_N$ is a symmetric function of λ_j^C; consider its dependence on λ_1^C. Using the commutation relations (1.3) one obtains after some algebra the explicit dependence of $\mathscr{F}_N$ on λ_1^C as follows:

$$\begin{aligned}\mathscr{F}_N(\{\lambda^C\}_{N-1},\{\lambda^B\}_N)&=\left\langle 0\left|\prod_{j=2}^{N-1}\mathbb{C}(\lambda_j^C)\psi(0)\mathbb{C}(\lambda_1^C)\prod_{j=1}^{N}\mathbb{B}(\lambda_j^B)\right|0\right\rangle\\&=\sum_{n=1}^{N} g(\lambda_1^C,\lambda_n^B)\left(r(\lambda_1^C)\prod_{j\neq n} f(\lambda_j^B,\lambda_n^B)f(\lambda_1^C,\lambda_j^B)-r(\lambda_n^B)\right.\\&\quad\left.\times\prod_{j\neq n} f(\lambda_j^B,\lambda_1^C)f(\lambda_n^B,\lambda_j^B)\right)\left\langle 0\left|\prod_{j=2}^{N-1}\mathbb{C}(\lambda_j^C)\psi(0)\prod_{j\neq n}^{N}\mathbb{B}(\lambda_j^B)\right|0\right\rangle.\end{aligned} \tag{A2.2}$$

Due to the Bethe equations (1.12) functions $r(\lambda_n^B)$ and $r(\lambda_1^C)$ are expressed in terms of λ_j^B, λ_j^C; this results in

$$\begin{aligned}\mathscr{F}_N(\{\lambda^C\}_{N-1},\{\lambda^B\}_N)&=\sum_{n=1}^{N} g(\lambda_1^C,\lambda_n^B)\prod_{j\neq n} f(\lambda_j^B,\lambda_n^B)\\&\quad\times\left(\prod_{j\neq 1}(f(\lambda_j^C,\lambda_1^C)/f(\lambda_1^C,\lambda_j^C))\prod_{j\neq n} f(\lambda_1^C,\lambda_j^B)-\prod_{j\neq n} f(\lambda_j^B,\lambda_1^C)\right)\\&\quad\times\left\langle 0\left|\prod_{j\neq 1}^{N-1}\mathbb{C}(\lambda_j^C)\psi(0)\prod_{j\neq n}^{N}\mathbb{B}(\lambda_j^B)\right|0\right\rangle.\end{aligned} \tag{A2.3}$$

Now the $(\lambda_1^C)^{-2}$ asymptotical behaviour is quite evident. The asymptotics in λ_1^B is established in a similar way.

So $\mathscr{P}_N$ is a polynomial of degree $N-2$ in each λ. Returning to (A2.1) one concludes that this formula permits the determination of the values of $\mathscr{P}_N$ (considered as a polynomial in λ_1^B) at $(N-1)$ points λ_j^C $(j=1,\ldots,N-1)$ provided that $\mathscr{P}_{N-1}$ is known. This implies that the polynomial $\mathscr{P}_N$ can be restored by means of the Lagrange formula. Thus the rule of the recursive calculation of $\mathscr{P}_N$ is given.

Appendix 3

Using (4.10) and (4.11) one arrives at the following expression for $\mathscr{F}_N$:

$$\mathscr{F}_N = \sum_{\{\lambda^B\}=\bigcup_{\alpha=0}^A\{\lambda_\alpha^B\}} \sum_{\{\lambda^C\}=\bigcup_{\alpha=0}^A\{\lambda_\alpha^C\}} \prod_{\alpha<\beta} l_{\beta,\beta+1}(\lambda_\alpha^B)\prod_{\lambda_\beta^B} f(\lambda_\alpha^B,\lambda_\beta^B)$$
$$\times \prod_{\alpha<\beta}\prod_{\lambda_\beta^C} l_{\alpha,\alpha+1}(\lambda_\beta^C)\prod_{\lambda_\alpha^C} f(\lambda_\beta^C,\lambda_\alpha^C)\prod_{\alpha=0}^{A}\exp(\theta_\alpha N_{\alpha,\alpha+1}^B)$$
$$\times\left\langle 0\left|\prod_{\lambda_\alpha^C}\mathbb{C}_{\alpha,\alpha+1}(\lambda_\alpha^C)\varphi_{a_\alpha}(x_\alpha)\prod_{\lambda_\alpha^B}\mathbb{B}_{\alpha,\alpha+1}(\lambda_\alpha^B)\right|0\right\rangle. \qquad \text{(A3.1)}$$

Below we give the main points of two different proofs of the formula (4.12).

The first proof is based on the analysis of singularities of the right-hand sides of (4.12) and (A3.1). The structure of the singularities and the asymptotical behaviour prove to be the same. The most important are the first-order poles at $\lambda_j^C\to\lambda_k^B$. As $\lambda_1^C\to\lambda_1^B$, one has for (A3.1) (other possibilities are easily restored due to the symmetry in λ_j^C and in λ_k^B) that

$$\mathscr{F}(\{\lambda^C\}_{N_C},\{\lambda^B\}_{N_B},\{l_{\beta,\beta+1}^C\}_{N_C},\{l_{\beta,\beta+1}^B\}_{N_B})\to \mathrm{i}c(\lambda_1^B-\lambda_1^C)^{-1}\sum_{\gamma=0}^{A}\prod_{\alpha<\gamma} l_{\alpha,\alpha+1}(\lambda_1^C)$$
$$\times\prod_{\beta>\gamma} l_{\beta,\beta+1}(\lambda_1^B)\exp(\theta_\gamma)(l_{\gamma,\gamma+1}(\lambda_1^B)$$
$$-l_{\gamma,\gamma+1}(\lambda_1^C))\prod_{j\neq1} f(\lambda_1^C,\lambda_j^C)f(\lambda_j^B,\lambda_1^B)$$
$$\times\mathscr{F}(\{\lambda^C\}_{N_C-1},\{\lambda^B\}_{N_B-1},\{l_{\beta,\beta+1}^C\}_{N_C-1},\{l_{\beta,\beta+1}^B\}_{N_B-1})^{\mathrm{mod}(\lambda_1^B,\gamma)} \qquad \text{(A3.2)}$$

$$l_{\beta,\beta+1}^{\mathrm{mod}(\lambda,\gamma)}(\mu)=l_{\beta,\beta+1}(\mu)\qquad(\beta\neq\gamma)\qquad l_{\beta,\beta+1}^{\mathrm{mod}(\lambda,\gamma)}(\mu)=l_{\gamma,\gamma+1}(\mu)\frac{f(\mu,\lambda_1)}{f(\lambda_1,\mu)}.$$

Now the Bethe equations (1.12) must be taken into account. Putting those into the form

$$\prod_{\alpha=0}^{A} l_{\alpha,\alpha+1}(\lambda_j)=\prod_{k\neq j}\frac{f(\lambda_j,\lambda_k)}{f(\lambda_k,\lambda_j)} \qquad \text{(A3.3)}$$

one expresses the functional variable $l_{A,A+1}(\lambda_j)$ in terms of other variables $l_{\alpha,\alpha+1}(\lambda_j)$, $\alpha\neq A$. $\mathscr{F}$ can therefore be considered as independent of $l_{A,A+1}$. It makes possible arbitrary independent variations of spectral parameters λ_j^C, λ_k^B supposing that they nevertheless satisfy the Bethe equations and variables $l_{\alpha,\alpha+1}(\lambda_j)$ $(\alpha=1,\ldots,A-1)$ are kept constant (i.e. that only $l_{A,A+1}(\lambda_j)$ is changed which does not enter the expression for the form factor). This results in the reduction formula

$$\begin{aligned}\mathscr{F}(\{\lambda^C\}_{N_C},\{\lambda^B\}_{N_B},\{l^C_\beta\}^A_{\beta=1},\{l^B_\beta\}^A_{\beta=1}) &\to \mathrm{i}c(\lambda^B_1-\lambda^C_1)^{-1}\\ &\times\sum_{\gamma=0}^{A}\exp(\theta_\gamma)(l_\gamma(\lambda^C_1)l_\gamma^{-1}(\lambda^B_1)\\ &-l_{\gamma+1}(\lambda^C_1)l^{-1}_{\gamma+1}(\lambda^B_1))\prod_{j\neq1}f(\lambda^B_j,\lambda^B_1)f(\lambda^C_1,\lambda^C_j)\\ &\times\mathscr{F}(\{\lambda^C\}_{N_C-1},\{\lambda^B\}_{N_B-1},\{l^C_\beta\}^A_{\beta=1},\{l^B_\beta\}^A_{\beta=1})^{\mathrm{mod}(\lambda^B_1,\gamma)}\\ &+\mathrm{i}c(\lambda^B_1-\lambda^C_1)^{-1}(l_A(\lambda^C_1)l_A^{-1}(\lambda^B_1)-\prod_{j\neq1}\frac{f(\lambda^C_j,\lambda^C_1)f(\lambda^B_1,\lambda^B_j)}{f(\lambda^C_1,\lambda^C_j)f(\lambda^B_j,\lambda^B_1)}\\ &\times\prod_{j\neq1}f(\lambda^B_j,\lambda^B_1)\prod_{j\neq1}f(\lambda^C_1,\lambda^C_j)\\ &\times\mathscr{F}(\{\lambda^C\}_{N_C-1},\{\lambda^B\}_{N_B-1},\{l^C_\beta\}^A_{\beta=1},\{l^B_\beta\}^A_{\beta=1})^{\mathrm{mod}(\lambda^B_1,A+1)}.\end{aligned}\tag{A3.4}$$

Here

$$l_0(\lambda)\equiv1\qquad l_\beta(\lambda)=\prod_{\alpha<\beta}l_{\alpha,\alpha+1}(\lambda)\qquad l_\beta(\mu)^{\mathrm{mod}(\lambda,\gamma)}=l_\beta(\mu)\qquad(\beta\leqslant\gamma)$$

$$l_\beta(\mu)^{\mathrm{mod}(\lambda,\gamma)}=l_\beta(\mu)f(\mu,\lambda)/f(\lambda,\mu)\qquad(\beta>\gamma).$$

To prove (4.12) one first establishes that the singularities of the right-hand side calculated by means of (4.13)-(4.15) are in agreement with (A3.4). Then the proof is quite similar to the one for the current form factor given in [4].

The second way to establish the validity of (4.12) uses the direct resummation over subpartitions in (A3.1). The proof is given below in the important particular case $A_+=A_-=1$, $A_0=0$ (the general case is considered similarly). The proof is made in three steps.

(i) Using the formula (1.10) for scalar products one has from (A3.4):

$$\begin{aligned}\mathscr{F}=\sum_{\mathrm{part}_B}\sum_{\mathrm{part}_C}&l_{12}(\lambda^B_2)f(\lambda^B_2,\lambda^B_1)l_{01}(\lambda^B_2)f(\lambda^B_2,\lambda^B_0)\\ &\times l_{01}(\lambda^C_2)f(\lambda^C_2,\lambda^C_0)f(\lambda^C_1,\lambda^C_2)l_{12}(\lambda^B_0)f(\lambda^B_0,\lambda^B_1)l_{01}(\lambda^C_1)\\ &\times f(\lambda^C_1,\lambda^C_0)l_{01}(\lambda^A_0)f(\lambda^{AC}_0,\lambda^{DC}_0)f(\lambda^{AB}_0,\lambda^{DB}_0)Z(\{\lambda^{DB}_0\},\{\lambda^{AC}_0\})\\ &\times Z(\{\lambda^{DC}_0\},\{\lambda^{AB}_0\})l_{12}(\lambda^A_1)f(\lambda^{AC}_1,\lambda^{DC}_1)f(\lambda^{AB}_1,\lambda^{DB}_1)\\ &\times Z(\{\lambda^{DB}_1\},\{\lambda^{AC}_1\})Z(\{\lambda^{DC}_1\},\{\lambda^{AB}_1\}).\end{aligned}\tag{A3.5}$$

The sum here is taken over partitions part_B, part_C

$$\{\lambda^B\}=\{\lambda^{AB}_0\}\cup\{\lambda^{DB}_0\}\cup\{\lambda^{AB}_1\}\cup\{\lambda^{DB}_1\}\cup\{\lambda^B_2\}$$

$$\{\lambda^C\}=\{\lambda^{AC}_0\}\cup\{\lambda^{DC}_0\}\cup\{\lambda^{AC}_1\}\cup\{\lambda^{DC}_1\}\cup\{\lambda^C_2\}$$

where $\mathrm{card}\{\lambda^C_2\}=\mathrm{card}\{\lambda^B_2\}=1$. All the product signs here are omitted (which products must be taken is clear from (A3.1) and (1.10)). The notations

$$\{\lambda^A_\alpha\}=\{\lambda^{AB}_\alpha\}\cup\{\lambda^{AC}_\alpha\}\qquad\{\lambda^B_\alpha\}=\{\lambda^{AB}_\alpha\}\cup\{\lambda^{DB}_\alpha\}$$

are used.

(ii) Further, one expresses l_{12} in terms of l_{01} by means of the Bethe equations (1.12).

(iii) The following partition of λ^B, λ^C is made:

$$\{\lambda_{\rm I}^C\}=\{\lambda_0^{AC}\}\cup\{\lambda_1^{DC}\}\cup\{\lambda_2^C\} \qquad \{\lambda_{\rm I}^B\}=\{\lambda_0^{DB}\}\cup\{\lambda_1^{AB}\}$$

$$\{\lambda_{\rm II}^C\}=\{\lambda_1^{AC}\}\cup\{\lambda_0^{DC}\} \qquad \{\lambda_{\rm II}^B\}=\{\lambda_0^{AB}\}\cup\{\lambda_1^{DB}\}\cup\{\lambda_2^B\}$$

$$\begin{aligned}\mathrm{card}\{\lambda_{\rm I}^C\}&=\mathrm{card}\{\lambda_{\rm II}^B\}=\mathrm{card}\{\lambda_{\rm I}^B\}+1\\&=\mathrm{card}\{\lambda_{\rm II}^C\}+1.\end{aligned} \tag{A3.6}$$

Now one makes the resummation in (A3.5) according to the partition (A3.6) obtaining formula (4.12) for $A_+=A_-=1$, $A_0=0$

$$\begin{aligned}\mathcal{F}=&\sum_{\{\lambda^B\}_N=\{\lambda_{\rm I}^B\}_{n-1}\cup\{\lambda_{\rm II}^B\}_{N-n+1}}\;\sum_{\{\lambda^C\}_N=\{\lambda_{\rm I}^C\}_n\cup\{\lambda_{\rm II}^C\}_{N-n}}\\&\times l_{01}(\lambda_{\rm I}^C)l_{01}^{-1}(\lambda_{\rm I}^B)f(\lambda_{\rm II}^B,\lambda_{\rm I}^B)f(\lambda_{\rm I}^C,\lambda_{\rm II}^C)\\&\times\langle 0|\mathbb{C}(\lambda_{\rm I}^C)\psi^+(0)\mathbb{B}(\lambda_{\rm I}^B)|0\rangle\langle 0|\mathbb{C}(\lambda_{\rm II}^C)\psi(0)\mathbb{B}(\lambda_{\rm II}^B)|0\rangle.\end{aligned} \tag{A3.7}$$

Appendix 4

Let us show that formula (4.8) permits us to obtain the coordinate representation of the Bethe eigenvectors. To be more concrete, consider the case of the *XXX* Heisenberg magnetic chain with $A-1$ sites; the spin operator in the αth site is S_α. The corresponding parameters of the generalised model are

$$l_{\beta,\beta+1}(\lambda)=(\lambda+\mathrm{i}S_\beta)/(\lambda-\mathrm{i}S_\beta). \tag{A4.1}$$

Operators $\mathbb{B}_{\alpha,\alpha+1}(\lambda)$ in this model are proportional to the spin creation operators at the site α

$$\mathbb{B}_{\alpha,\alpha+1}(\lambda)=(\lambda-\mathrm{i}S_\alpha)^{-1}S_\alpha^-. \tag{A4.2}$$

Denote $n_\alpha=\mathrm{card}\{\lambda^\alpha\}$ in (4.8). Putting (A4.1) and (A4.2) into (4.8) one obtains

$$\begin{aligned}\prod_{j=1}^N\mathbb{B}(\lambda_j)|0\rangle=&\sum_{\{\lambda\}_N=\bigcup_{\alpha=0}^A\{\lambda^\alpha\}}\;\prod_{\alpha=0}^A\prod_{j=1}^{n_\alpha}(\lambda_j^\alpha-\mathrm{i}S_\alpha)^{-1}l_\alpha(\lambda_j^\alpha)\\&\times\prod_{\alpha=0}^A\prod_{\beta=\alpha+1}^A\prod_{j=1}^{n_\alpha}\prod_{k=1}^{n_\beta}f(\lambda_j^\alpha,\lambda_k^\beta)\prod_{\alpha=0}^A(S_\alpha^-)^{n_\alpha}|0\rangle.\end{aligned} \tag{A4.3}$$

For the homogeneous chain with spin S one can rewrite this formula as a sum over permutations:

$$\begin{aligned}\prod_{j=1}^N\mathbb{B}(\lambda_j)|0\rangle=&\prod_{j=1}^N(\lambda_j-\mathrm{i}S)\sum_{\alpha_1\leqslant\ldots\leqslant\alpha_N}\;\sum_P\prod_{j=1}^N\\&\times\left(\frac{\lambda_{P_j}+\mathrm{i}S}{\lambda_{P_j}-\mathrm{i}S}\right)^{N-\alpha_j}\prod_{k\geqslant j}f(\lambda_{P_j},\lambda_{P_k})\prod_{j=1}^N(n_j!)^{-1}(S_{\alpha_j}^-)^{n_j}|0\rangle.\end{aligned} \tag{A4.4}$$

Here n_j is the number of equal α_j. As $(S_\alpha^-)^2=0$ at $S=\frac{1}{2}$ one has $\alpha_1<\alpha_2<\ldots<\alpha_N$ in (A4.4), so in this case the well known formula of the coordinate Bethe ansatz is restored.

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